1. **Schwarzschild Geometry Basics, Part I.** Recall the three-dimensional spatial Schwarzschild metric \( g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{R}^3} \), defined on the manifold \( M \) given by \( M = \mathbb{R}^3 \setminus \{0\} \) for \( m > 0 \), \( M = \mathbb{R}^3 \) for \( m = 0 \), and \( M = \{x \in \mathbb{R}^3 : |x| > -\frac{m}{2}\} \) for \( m < 0 \).

   a. Find \( \text{Ric}(g_S) \), which does not vanish; you should observe that its trace \( R(g_S) \), the scalar curvature, does vanish.

   b. Show that
   \[
   m = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{|x|=r} \sum_{i,j=1}^3 ((g_S)_{i,j,i} - (g_S)_{ii,j}) v_i^j d\sigma_e
   \]
   where the computation is done in the coordinates \((x^1, x^2, x^3)\), and where \( \nu_e \) is the Euclidean outward unit normal, and \( d\sigma_e \) is the Euclidean area measure (where \((x^i)\) are Cartesian coordinates for the Euclidean metric).

   c. When \( m < 0 \), \( A(r) \to 0 \) as \( r \to -(\frac{m}{2})^+ \). Show that a radial geodesic from \( r = r_0 > -\frac{m}{2} \) to \( r = -\frac{m}{2} \) has finite length. Can the Schwarzschild metric with \( m < 0 \) be smoothly completed by adding in a point?

   d. Let \( m > 0 \). Find an isometric embedding of \((M, g_S)\) into Euclidean space \((\mathbb{R}^4, g_{\mathbb{R}^4})\), identified in Cartesian coordinates \((x, y, z, w)\) with \((\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)\). It might be easiest to use the other coordinates we introduced for the Schwarzschild metric: \((1 - \frac{2m}{r})^{-1}dr^2 + r^2 g_{\mathbb{R}^2}, r > 2m\). (This corresponds to “half” of \((M, g_S)\). The map you get will then extend by reflection to the other “half.”) For \( \omega \in \mathbb{S}^2 \), look for an embedding of the form \( x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4 \). Use this to sketch a picture of the Schwarzschild spatial slice.

   e. When \( m < 0 \) the argument breaks in part e. down. Instead, look for an isometric embedding into Minkowski space \( \mathbb{M}^4 \), which is identified with \( \mathbb{R}^4 \) with the metric \( dx^2 + dy^2 + dz^2 - dw^2 \).

2. **Schwarzschild Geometry Basics, Part II.** Let \( \nabla \) be the connection on \((M, g_S)\), and for vector fields \( X \) and \( Y \) tangent to a surface \( \Sigma \subset M \), let \( \Pi(X, Y) = (\nabla_X Y)^{\text{Nor}} \), and let \( \mathbf{H} = \text{tr}_\Sigma(\Pi) \).

   a. For \( m > 0 \), show that \( r \mapsto \frac{2m}{r^2} \) induces an isometry of \( g_S \) which fixes \( \Sigma_0 = \{r = \frac{m}{2}\} \).

   b. For \( m > 0 \), show that \( \Sigma_0 \) is totally geodesic in \( M \). Express \( m \) in terms of the area of \( \Sigma_0 \).

   c. Find the area \( A(r) \) of \( S_r = \{x : |x| = r\} \) of \( S_r \) in the metric \( g_S \). For \( m > 0 \), show that \( A(r) \) has a global minimum at \( r = \frac{m}{2} \).

   d. Fix \( r \) and find the second fundamental form and the mean curvature vector \( \mathbf{H} \) of \( S_r = \{x : |x| = r\} \) in the metric \( g_S \).

   e. Compare \( A'(r) \) to \( \int_{S_r} \mathbf{H} \cdot \mathbf{X} \ d\sigma \), where \( \mathbf{X} = \frac{\partial}{\partial r} \), and \( d\sigma \) is the area measure induced by \( g_S \).
f. For $m > 0$, show that there are no closed minimal surfaces in $(M, g_S)$ other than $\Sigma_0$, using an argument along the lines of the proof that there are no closed minimal surfaces in Euclidean space.

g. If $\nu_0$ is a unit normal to a surface with mean curvature vector $H$, let $H = \langle H, \nu_0 \rangle g_S$. The Hawking mass of a surface $\Sigma$ is given by

$$m_H(\Sigma) = \sqrt{A(\Sigma) \over 16\pi} \left(1 - {1 \over 16\pi} \int_\Sigma H^2 \, d\sigma_S \right).$$

Find $m_H(S_r)$.

3. SCHWARZSCHILD GEOMETRY BASICS, PART III. In Euclidean space, the spheres minimize surface area for a given enclosed volume $V$. In fact if a closed surface of area $A$ encloses a volume $V$, the isoperimetric inequality in three dimensions is $V \leq A^{3/2} / 6\sqrt{\pi}$.

Let $m > 0$. Hubert Bray showed that the spheres $S_r = \{x : |x| = r\}$ in $(M, (1 + m/2)g_S)$ are isoperimetric in the homology class of $\Sigma_0$ (defined above). In other words, amongst all surfaces homologous to $\Sigma_0$ and enclosing a certain volume $V$ with $\Sigma_0$, the one with smallest area is the sphere $S_r$ of the correct $r$ value to enclose volume $V$.

a. Show that the volume $V(r)$ enclosed by $\Sigma_0$ and $S_r$ ($r \geq m/2$) and $\Sigma$ has the expansion

$$V(r) = \frac{4\pi r^3}{3} \left(1 + \frac{9m}{2r} + O(mr^{-2})\right).$$

b. Conclude that the volume $V$ enclosed by $\Sigma_0$ and the sphere $S_r$ of area $A$ has the expansion

$$V(A) = \frac{A^{3/2}}{6\sqrt{\pi}} \left(1 + \frac{(3\sqrt{\pi})m}{\sqrt{A}} + O(mA^{-1})\right).$$

4. LINEARIZATION OF THE SCALAR CURVATURE MAP. Let $R(g) = g^{ij}R_{ij}$ be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation $g(\epsilon) = g + \epsilon h$ of $g$ in the direction of a symmetric $(0, 2)$-tensor field $h$ (more generally, note that all you will use is that $g(\epsilon)$ is a metric smooth in $t$, with $g(0) = g$ and $g'(0) = h$). Assume that for small $|\epsilon|$, $g(\epsilon)$ is a metric, as would be the case for $h$ compactly supported. Define $L_g(h) := DR_g(h) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} R(g(\epsilon))$.

a. Derive the scalar curvature formula

$$R(g) = g^{ij}R_{ij} = g^{ij} \left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{kl} \Gamma^l_{ij} - \Gamma^k_{jl} \Gamma^l_{ik}\right).$$

b. Verify that the difference $S(X, Y) := \nabla_X Y - \nabla_X Y$ defines a vector-valued $(0, 2)$-tensor (i.e. a $(1, 2)$ tensor $\tilde{S}(\theta, X, Y) = \theta(S(X, Y))$). Thus $\Gamma^k_{ij} := \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \Gamma^k_{ij}$ form the components $(\delta\Gamma)^k_{ij}$ of a $(1, 2)$-tensor $(\delta\Gamma)$. Argue that $\tilde{\Gamma}^k_{ij} = \frac{1}{2} g^{km}(h_{mj,i} + h_{im,j} - h_{ij,m})$, where the covariant derivative is
taken with respect to \( g(0) \). (Hint: use \( g(0) \)-normal coordinates at \( p \).)

c. Use the preceding part to aid in verifying the identities \( \frac{d}{dt} \bigg|_{t=0} R_{ij} = (\delta \Gamma)^k_{ij,k} - (\delta \Gamma)^k_{ik,j} \), and then

\[
L_g(h) = -\Delta_g(\text{tr}_g(h)) + \text{div}_g(\text{div}_g(h)) - \langle h, \text{Ric}(g) \rangle_g
\]

where the inner product of two \((0,2)\)-tensors \( S \) and \( T \) is given by \( \langle S, T \rangle = S_{ij} T_{k\ell} g^{ik} g^{j\ell} \), for example \( \text{tr}_g(S) = \langle g, S \rangle \).

d. Show that \( L_g^* N = (\Delta_g N + \text{Hess}_g N - N \text{Ric}(g)) \), by integrating \( \int_M N L_g(h) \ dv_g \) by parts (for \( h \) compactly supported away from the boundary of \( M \)).

e. Show directly (and in one line) that if \( h \) is symmetric with compact support, and if \( L_g h \geq 0 \), then \( L_g h = 0 \).

f. Show by elementary methods that there exists an infinite-dimensional space of symmetric TT tensors (trace-free, divergence-free) on \((\mathbb{R}^3, g_E)\) with compact support. Such tensors automatically satisfy \( L_g h = 0 \).

**Problem 5. Static potentials, I.** Suppose \((M, g)\) is Riemannian.

a. Suppose that \( L_g^* N = 0 \), and that \( \gamma \) is a unit-speed geodesic in \((M^n, g)\). Let \( h(t) = N(\gamma(t)) \). Prove that \( h(t) \) satisfies a second-order linear ODE. What does this say about the dimension of the kernel of \( L_g^* \)?

b. Suppose that \( L_g^* N = 0 \), but that \( N \) is not identically zero. Show that \( \Sigma = N^{-1}(0) \) is a regular hypersurface, which is totally geodesic (zero second fundamental form). Hint: If \( p \in \Sigma \) and \( dN_p = 0 \), what does part a. have to say about things?

c. Suppose that \((M^n, g)\) is a closed manifold with negative scalar curvature. Find the kernel of \( L_g^* \).

d. Consider the metric \( g = (n - 2)^{-1} g_{S^1} \oplus g_{S^{n-1}} \) on \( S^1 \times S^{n-1} \). Show that \( N(t, \omega) = \sin t \) solves \( L_g^* N = 0 \).

e. Does every Ricci-flat metric have a nontrivial element \( N \) in the kernel of \( L_g^* \)? What can you say in case a metric \((M, g)\) on a closed manifold with zero scalar curvature admits a nontrivial \( N \) with \( L_g^* N = 0 \)?

f. Let \( N : M \to \mathbb{R} \) be a smooth function. Define the Lorentzian metric \( \bar{g} = -N^2 dt^2 \oplus g \) on the space \( S = I \times \{ p \in M : N(p) \neq 0 \} \). Prove that for \( X, Y \) tangent to \( M \) at \( p \) with \( N(p) \neq 0 \), we have \( \text{Ric}(\bar{g})(X, Y) = \text{Ric}(g)(X, Y) - \frac{1}{N(p)} \text{Hess}_g N(p), \text{Ric}(\bar{g})(X, \frac{\partial}{\partial t}) = 0, \text{and } \text{Ric}(\bar{g})(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = N \Delta_g N \).

g. Conclude from part a. that a function \( N \) on \( M \) is a nontrivial element of the kernel of \( L_g^* \) if and only if the metric \( \bar{g} \) as above is an Einstein metric. (Note that in the preceding problem you said something about the set \( \{ p \in M : N(p) = 0 \} \) where the metric \( \bar{g} \) may have issues.)

5. Conformal changes of metric.
a. Suppose \((M^n, g)\) is a Riemannian metric and \(\hat{g} = e^{\varphi}g\). Show that
\[
R(\hat{g}) = e^{-\varphi} \left( R(g) - (n - 1)\Delta_g \varphi - \frac{1}{4}(n - 1)(n - 2)|\nabla \varphi|_g^2 \right).
\]
b. In case \(n \geq 3\), if we write \(e^{\varphi} = u^{\frac{4}{n-2}}\) for \(u > 0\), then
\[
R(\hat{g}) = u^{-\frac{n+2}{n-2}} \left( R(g)u - \frac{4(n-1)}{(n-2)} \Delta_g u \right).
\]
c. Suppose \(M\) is compact with empty boundary. Let \(c(n) = \frac{n-2}{4(n-1)}\). Let \(L_g u = \Delta_g u - c(n)R(g)u\), the conformal Laplacian. Show that the total scalar curvature of \(\hat{g} = u^{\frac{4}{n-2}}g\) is given by
\[
\int_M R(\hat{g}) \, dv_{\hat{g}} = c(n)^{-1} \int_M (|\nabla u|_{\hat{g}}^2 + c(n)R(g)u^2) \, dv_g.
\]
HINT: Show that \(dv_{\hat{g}} = u^{\frac{2n}{n-2}} \, dv_g\).

6. SOME ASYMPTOTIC EXPANSIONS. Suppose \((\mathbb{R}^3 \setminus B_{r_0}(0), g)\) is harmonically flat: \(g = u^4 g_E\), \(R(g) = 0\), i.e. \(\Delta_g u = 0\), with \(u(x) \to 1\) as \(|x| \to +\infty\). We saw the expansion \(u(x) = 1 + \frac{A}{|x|} + \frac{\beta_i x^i}{|x|^2} + O(|x|^{-3})\) via spherical harmonics.

a. Let \(x = y + c\), for \(c \in \mathbb{R}^3\). For \(|y + c| > r_0\), find the asymptotic expansion of \(u\) as a function of \(y\). Show for \(A \neq 0\) that there is a unique choice of \(c \in \mathbb{R}^3\) for which \(\tilde{u}(y) := u(y+c) = 1 + \frac{A}{|y|} + O(|y|^{-3})\).

b. Compute \(\lim_{r \to +\infty} \int_{|x| = r} x^k \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j}) \nu^i \, d\sigma_e\) where \(\nu^i = \frac{x^i}{r}\). (Warning: this gives the center of mass, but the flux integral isn’t the right form for more general asymptotically flat metrics.)

7. CONSTRAINTS MAP IN HARMONIC ASYMPTOTICS. Define the operator \((\tilde{\mathcal{L}}_g(X))_{ij} = X_{ij}^k + X_{j,i}^k - X_{ik}^j g_{ij}\). If \(\gamma\) is a metric on \(M^3\), let \(g = u^4 \gamma\) and \(\pi_{ij} = u^2 (\mathcal{L}_\gamma(X))_{ij}\) for \(u > 0\).

a. Compute the constraints map \(\Phi(g, \pi) = (R(g) - |\pi|^2_\gamma + \frac{1}{2} (\text{tr} \pi)^2, \text{div} \pi, \text{tr} \pi)\), and in case \(\gamma = g_E\), show that the vacuum constraints \(\Phi(g, \pi) = 0\) can be written, in a Cartesian coordinate system for the background \(g_E\), as follows (subscripts for the flat metric omitted):
\[
8\Delta u = u \left( -|\tilde{\mathcal{L}} X|^2 + \frac{1}{2} (\text{tr} (\tilde{\mathcal{L}} X))^2 \right)
\]
\[
\Delta X^i + 4u^{-1} u_{,i} (\tilde{\mathcal{L}} X)_i^j - 2u^{-1} u_{,i} \text{tr} (\tilde{\mathcal{L}} X) = 0
\]
b. If the above equations in part a. hold on an asymptotic end of an AF manifold \((M, g)\), one can show that \(u\) and \(X\) have partial expansions \(u(x) = 1 + \frac{A}{|x|} + O(|x|^{-2})\), \(X^i(x) = \frac{B^i}{|x|} + O(|x|^{-2})\), along with fall off for derivatives. Show that \(\pi_{ij} = -\frac{B^i x_j + B^j x_i}{|x|^3} + \sum_k \frac{B^k x^j x^i}{|x|^3} \delta_{ij} + O(|x|^{-3})\), and that \(P^i = -\frac{B^i}{2}\) is the ADM linear momentum.

8. Assume that \(h\) is a (smooth) transverse-traceless tensor at the Euclidean metric on \(\mathbb{R}^3\). Let’s use Cartesian coordinates \(x\), so that covariant derivative components are computed via partial derivatives (the Christoffel symbols vanish). So \(0 = \text{tr} g_E h = \sum_{i=1}^3 h_{ii},\) and \(0 = (\text{div} g_E h)_j = \sum_{i=1}^3 h_{ij,j}.\) Now,
assume that $h$ has compact support. Let $\gamma_\epsilon = g_{\mathbb{R}^n} + \epsilon h$, and for $|\epsilon|$ sufficiently small, let $u_\epsilon > 0$ be the associated conformal factor so that with $g_\epsilon = u_\epsilon^4 \gamma_\epsilon$, $R(g_\epsilon) = 0$, and $u_\epsilon$ tends to 1 at infinity. Near infinity each $u_\epsilon$ is harmonic, and as such has an asymptotic expansion $u_\epsilon = 1 + \frac{m(\epsilon)}{|x|^2} + O(|x|^{-2})$.

a. Show that $16\pi m(\epsilon) = -\int_{\mathbb{R}^3} R(\gamma_\epsilon) u_\epsilon \, dv_{g_\epsilon}$.

b. Show that $m'(0) = 0$ and that $16\pi m''(0) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_{g_\epsilon} h|^2 \, dv_{g_\epsilon}$.

9. Rigidity computations for PET. Suppose $(M, g)$ is asymptotically flat with one end, with $R(g) \geq 0$, $R(g) \in L^1(M)$, and with ADM mass $m = 0$.

a. One can show that there is a conformal factor $u > 0$, $u \to 1$ near infinity, with $u(x) = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{-n+1})$, so that $R(u^{\frac{4}{n-2}} g) = 0$. Argue that $A \leq 0$, and that the mass of $u^{\frac{4}{n-2}} g$ is $m + 2A \leq m$. Show that if $R(g)$ does not vanish identically, then $A < 0$.

b. Suppose $m = 0$. By part a., we have $R(g) = 0$. We want to show the Ricci curvature vanishes. Let $0 \leq \zeta_\theta \leq 1$ be a compactly-supported bump function which is identically 1 on the compact core of $M$, out to $|x| \leq \theta$, and $\zeta_\theta = 0$ outside $|x| \geq 2\theta$ (or $|x| \geq \theta + 1$, say). Let $h_\theta = \zeta_\theta \text{Ric}(g)$. Let $\gamma_\epsilon = g + \epsilon h_\theta$. Let $u_\epsilon > 0$ be so that $R(u_\epsilon^{\frac{4}{n-2}} \gamma_\epsilon) = 0$, with $u_\epsilon \to 1$ at infinity. (Such a function exists for small $\epsilon$, because $\Delta_g - \frac{n-2}{4(n-1)} R(\gamma_\epsilon)$ is a small perturbation of $\Delta_g$, which is invertible in suitable weighted function spaces.) Let $g_\epsilon = u_\epsilon^{\frac{4}{n-2}} \gamma_\epsilon$, and let $m(\epsilon)$ be the ADM mass of $g_\epsilon$. Use the idea of #8a., and argue that $m'(0) = 0$. Then compute $m'(0)$ using the linearization of scalar curvature operator (#4c.) $DR_g(h) = -\Delta_g (\text{tr}_g h) + \text{div}_g \text{div}_g h - (h, \text{Ric}(g))_g$.

c. From part b., we have an complete manifold $(M, g)$ with vanishing Ricci curvature, which is also asymptotically flat. Use the Bishop-Gromov volume comparison to argue that $(M, g)$ must be isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$.

10. This problem refers to Proposition 3.2 in the Corvino-Pollack article. The proof as written has a gap. It was first written for $R(g) = 0$ or small, but the point of the way it is stated is to allow more general $R(g) \geq 0$. Locate the error. Then fix it! To do so, note that what is small is $R(g_\theta) - \psi_\theta R(g)$ for large $\theta$. Modify the desired scalar curvature of $R(u^4 g_\theta)$ to make the resulting PDE to solve for $u$ to be much nicer, and then complete the proof.

APPENDIX: Problems on Euclidean Harmonic Functions.

1. a. Verify that the following distributional equations hold: $\Delta(\frac{1}{2\pi} \log |x|) = \delta_0$ in dimension $n = 2$, while $\Delta(\frac{1}{(2-n)n\omega_n} |x|^{2-n}) = \delta_0$ in dimensions $n > 2$. Here $\delta_0$ is the Dirac delta distribution at the origin.

b. Suppose $f \in C^2_c(\mathbb{R}^n)$, $n > 2$. Suppose $\text{spt}(f) \subset \{x : |x| \leq K\}$. Then if we let $u(x) = \frac{1}{(2-n)n\omega_n} \int_{\mathbb{R}^n} |x - y|^{2-n} f(y) \, dy$, then $\Delta u = f$ by the above. Moreover, show that $u$ has an expansion of the form $u(x) = \frac{A}{|x|^{n-2}} + \frac{B_i x_i}{|x|^n} + O(|x|^{-n})$. Express the constants $A$ and $B_i$ in terms of integrals involving $f$. 


2. a. Show that if \( u \) is harmonic with an isolated singularity at \( x = 0 \), then the singularity is in fact removable if \( \lim_{x \to 0} |x|^{n-2} u(x) = 0 \) in case \( n > 2 \), and in case \( n = 2 \), if \( \lim_{x \to 0} \frac{u(x)}{\log |x|} = 0 \).

b. If \( K[u] \) is the Kelvin transform of \( u \), find \( \Delta(K[u]) \) in terms of \( \Delta u \). Conclude that \( K[u] \) is harmonic if and only if \( u \) is harmonic. Recall \( K[u](x) = |x|^{2-n} u(x^*) \), \( x^* = |x|^{-2} x \).

c. Prove that if \( n > 2 \) and \( u \) is harmonic near infinity. Prove that \( u \) is harmonic at infinity if and only if \( \lim_{|x| \to +\infty} u(x) = 0 \).

3. If \( v \) is harmonic at infinity and \( n > 2 \), \( v \) admits an expansion at infinity in terms of spherical harmonics. We derived the first two terms which give \( v(x) = \frac{a_0}{|x|^{n-2}} + \frac{a_i x^i}{|x|^{n}} + O(|x|^n) \). Derive the next order term, in case \( n = 3 \).

4. Let \( (S^n, g_0) \) be the standard unit round sphere, \( S^n \) embedded in \( \mathbb{R}^{n+1} \) as \( \{|x| = 1\} \). It is a fact that the lowest positive eigenvalue \( \lambda_1 \) for \( \Delta g_0 \) corresponds to the eigenfunctions \( x^i \) (Euclidean coordinates) restricted to the sphere. Compute \( \lambda_1 = n \) by using \( \Delta g_0(x^i) = -\lambda_1 x^i \). Multiply by \( x^i \), integrate by parts, and use the fact that \( \nabla g_0 x^i \) is the tangential component of \( \nabla x^i = e_i = \frac{\partial}{\partial x^i} \).

5. Recall Bôcher’s Theorem: if \( u > 0 \) is harmonic in a punctured ball \( B \setminus \{0\} \), there exist \( v \) harmonic in \( B \) and \( b \geq 0 \) so that \( u(x) = \begin{cases} b \log \left( \frac{1}{|x|} \right) + v(x), & n = 2 \\ b|x|^{2-n} + v(x), & n > 2. \end{cases} \)

a. Show that \( b \) and \( v \) are uniquely determined.

b. \( \Omega \subset \mathbb{R}^n \) is an open set, \( n > 2 \). If \( u \) is harmonic in \( \Omega \setminus \{a\} \) \( (a \in \Omega) \), so that \( u > 0 \) in a deleted neighborhood of \( a \), show there is a number \( b \geq 0 \) and a function \( v \) harmonic on all of \( \Omega \) so that on \( \Omega \setminus \{a\}, u(x) = b|x-a|^{2-n} + v(x) \).

c. \( n > 2 \). If \( u \) is harmonic on \( B \setminus \{0\} \), and \( \lim \inf_{x \to 0} |x|^{n-2} u(x) > -\infty \), there exists \( v \) harmonic in \( B \), \( b \in \mathbb{R} \) so that \( u(x) = b|x|^{2-n} + v(x) \) on \( B \setminus \{0\} \).

d. What can you say about a positive harmonic function on \( \mathbb{R}^n \setminus \{0, a\}, a \neq 0 \)?