Summer School Problems on Geometry and Relativity

1. Schwarzschild Geometry Basics, Part I. Recall the three-dimensional spatial Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{E}}$, defined on the manifold M given by $M = \mathbb{R}^3 \setminus \{0\}$ for $m > 0$, $M = \mathbb{R}^3$ for $m = 0$, and $M = \{x \in \mathbb{R}^3 : |x| > -\frac{m}{2}\}$ for $m < 0$.

a. Find $\mathrm{Ric}(g_S)$, which does not vanish; you should observe that its trace $R(g_S)$, the scalar curvature, does vanish.

b. Show that

$$m = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{|x|=r} \sum_{i,j=1}^{3} \left((g_S)_{ij,i} - (g_S)_{ii,j}\right) \nu_e^j \, d\sigma_e$$

where the computation is done in the coordinates (x^1, x^2, x^3), and where ν_e is the Euclidean outward unit normal, and $d\sigma_e$ is the Euclidean area measure (where (x^i) are Cartesian coordinates for the Euclidean metric).

c. When $m < 0$, $A(r) \to 0$ as $r \to -(\frac{m}{2})^+$. Show that a radial geodesic from $r = r_0 > -\frac{m}{2}$ to $r = -\frac{m}{2}$ has finite length. Can the Schwarzschild metric with $m < 0$ be smoothly completed by adding in a point?

d. Let $m > 0$. Find an isometric embedding of (M, g_S) into Euclidean space $(\mathbb{R}^4, g_{\mathbb{E}})$, identified in Cartesian coordinates (x, y, z, w) with $(\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)$. It might be easiest to use the other coordinates we introduced for the Schwarzschild metric: $(1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{\mathbb{S}^2}$, $r > 2m$. (This corresponds to "half" of (M, g_S). The map you get will then extend by reflection to the other "half.") For $\omega \in \mathbb{S}^2$, look for an embedding of the form $x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Use this to sketch a picture of the Schwarzschild spatial slice.

e. When $m < 0$ the argument breaks in part e. down. Instead, look for an isometric embedding into Minkowski space $\mathbb{M}^4$, which is identified with $\mathbb{R}^4$ with the metric $dx^2 + dy^2 + dx^2 - dw^2$.

2. Schwarzschild Geometry Basics, Part II. Let ∇ be the connection on (M, g_S), and for vector fields X and Y tangent to a surface $\Sigma \subset M$, let $\mathbb{II}(X, Y) = (\nabla_X Y)^{\mathrm{Nor}}$, and let $\mathbf{H} = \mathrm{tr}_\Sigma(\mathbb{II})$.

a. For $m > 0$, show that $r \mapsto \frac{m^2}{4r}$ induces an isometry of g_S which fixes $\Sigma_0 = \{r = \frac{m}{2}\}$.

b. For $m > 0$, show that Σ_0 is totally geodesic in M. Express m in terms of the area of Σ_0.

c. Find the area $A(r)$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S. For $m > 0$, show that $A(r)$ has a global minimum at $r = \frac{m}{2}$.

d. Fix r and find the second fundamental form and the mean curvature vector $\mathbf{H}$ of $S_r = \{x : |x| = r\}$ in the metric g_S.

e. Compare $A'(r)$ to $\int_{S_r} \mathbf{H} \cdot \mathbf{X} \, d\sigma$, where $\mathbf{X} = \frac{\partial}{\partial r}$, and $d\sigma_S$ is the area measure induced by g_S.

f. For $m > 0$, show that there are no closed minimal surfaces in (M, g_S) other than Σ_0, using an argument along the lines of the proof that there are no closed minimal surfaces in Euclidean space.

g. If ν_S is a unit normal to a surface with mean curvature vector $\mathbf{H}$, let $H = \langle \mathbf{H}, \nu_S \rangle_{g_S}$. The *Hawking mass* of a surface Σ is given by

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\sigma_S \right).$$

Find $m_H(S_r)$.

g. If you consider the higher-dimensional Riemannian Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_{\mathbb{E}}$ with $m > 0$, find the area profile $A(r)$ of $S_r = \{x : |x| = r\}$, and find the radius at which $A(r)$ has a minimum. Compute the mass m in terms of this area.

3. Schwarzschild Geometry Basics, Part III. . In Euclidean space, the spheres minimize surface area for a given enclosed volume V. In fact if a closed surface of area A encloses a volume V, the *isoperimetric inequality* in three dimensions is $V \leq \frac{A^{3/2}}{6\sqrt{\pi}}$.

Let $m > 0$. Hubert Bray showed that the spheres $S_r = \{x : |x| = r\}$ in $(M, (1 + \frac{m}{2r})^4 g_{\mathbb{E}})$ are isoperimetric in the homology class of Σ_0 (defined above). In other words, amongst all surfaces homologous to Σ_0 and enclosing a certain volume V with Σ_0, the one with smallest area is the sphere S_r of the correct r value to enclose volume V.

a. Show that the volume $V(r)$ enclosed by Σ_0 and S_r $(r \geq \frac{m}{2})$ and Σ has the expansion

$$V(r) = \frac{4\pi r^3}{3} \left(1 + \frac{9m}{2r} + O(mr^{-2}) \right).$$

b. Conclude that the volume V enclosed by Σ_0 and the sphere S_r of area A has the expansion

$$V(A) = \frac{A^{3/2}}{6\sqrt{\pi}} \left(1 + \frac{(3\sqrt{\pi})m}{\sqrt{A}} + O(mA^{-1}) \right).$$

4. Linearization of the scalar curvature map. Let $R(g) = g^{ij} R_{ij}$ be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation $g(\epsilon) = g + \epsilon h$ of g in the direction of a symmetric $(0,2)$-tensor field h (more generally, note that all you will use is that $g(\epsilon)$ is a metric smooth in t, with $g(0) = g$ and $g'(0) = h$). Assume that for small $|\epsilon|$, $g(\epsilon)$ is a metric, as would be the case for h compactly supported. Define $L_g(h) := DR_g(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R(g(\epsilon))$.

a. Derive the scalar curvature formula

$$R(g) = g^{ij} R_{ij} = g^{ij} \left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik} \right).$$

b. Verify that the difference $S(X,Y) := \nabla_X Y - \widetilde{\nabla}_X Y$ defines a vector-valued $(0,2)$-tensor (i.e. a $(1,2)$ tensor $\widehat{S}(\theta, X, Y) = \theta(S(X,Y))$). Thus $\dot{\Gamma}^k_{ij} := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Gamma^k_{ij}$ form the components $(\delta\Gamma)^k_{ij}$ of a $(1,2)$-tensor $(\delta\Gamma)$. Argue that $\dot{\Gamma}^k_{ij} = \frac{1}{2} g^{km}(h_{mj;i} + h_{im;j} - h_{ij;m})$, where the covariant derivative is taken with respect to $g(0)$. (Hint: use $g(0)$-normal coordinates at p.)

c. Use the preceding part to aid in verifying the identities $\frac{d}{d\epsilon}\Big|_{\epsilon=0} R_{ij} = (\delta\Gamma)^k_{ij;k} - (\delta\Gamma)^k_{ik;j}$, and then

$$L_g(h) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - \langle h, \mathrm{Ric}(g)\rangle_g$$

where the inner product of two $(0,2)$-tensors S and T is given by $\langle S, T\rangle = S_{ij}T_{k\ell}g^{ik}g^{j\ell}$, for example $\mathrm{tr}_g(S) = \langle g, S\rangle$.

d. Show that $L_g^* N = -(\Delta_g N)g + \mathrm{Hess}_g N - N\mathrm{Ric}(g)$, by integrating $\int_M N L_g(h)\, dv_g$ by parts (for h compactly supported away from the boundary of M).

e. Show directly (and in one line) that if h is symmetric with compact support, and if $L_{g_{\mathbb{E}}} h \geq 0$, then $L_{g_{\mathbb{E}}} h = 0$.

f. Show by elementary methods that there exists an infinite-dimensional space of symmetric TT tensors (trace-free, divergence-free) on $(\mathbb{R}^3, g_{\mathbb{E}})$ *with compact support.* Such tensors automatically satisfy $L_{g_{\mathbb{E}}} h = 0$.

Problem 5. Static potentials, I. Suppose (M, g) is Riemannian.

a. Suppose that $L_g^* N = 0$, and that γ is a unit-speed geodesic in (M^n, g). Let $h(t) = N(\gamma(t))$. Prove that $h(t)$ satisfies a second-order linear ODE. What does this say about the dimension of the kernel of L_g^*?

b. Suppose that $L_g^* N = 0$, but that N is not identically zero. Show that $\Sigma = N^{-1}(0)$ is a regular hypersurface, which is totally geodesic (zero second fundamental form). Hint: If $p \in \Sigma$ and $dN_p = 0$, what does part a. have to say about things?

c. Suppose that (M^n, g) is a closed manifold with negative scalar curvature. Find the kernel of L_g^*.

d. Consider the metric $g = (n-2)^{-1} g_{\mathbb{S}^1} \oplus g_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Show that $N(t, \omega) = \sin t$ solves $L_g^* N = 0$.

e. Does every Ricci-flat metric have a nontrivial element N in the kernel of L_g^*? What can you say in case a metric (M, g) on a closed manifold with zero scalar curvature admits a nontrivial N with $L_g^* N = 0$?

f. Let $N : M \to \mathbb{R}$ be a smooth function. Define the Lorentzian metric $\bar{g} = -N^2 dt^2 \oplus g$ on the space $\mathcal{S} = I \times \{p \in M : N(p) \neq 0\}$. Prove that for X, Y tangent to M at p with $N(p) \neq 0$, we have $\mathrm{Ric}(\bar{g})(X, Y) = \mathrm{Ric}(g)(X, Y) - \frac{1}{N(p)}\mathrm{Hess}_g N(p)$, $\mathrm{Ric}(\bar{g})(X, \frac{\partial}{\partial t}) = 0$, and $\mathrm{Ric}(\bar{g})(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = N\Delta_g N$.

g. Conclude from part a. that a function N on M is a nontrivial element of the kernel of L_g^* if and only if the metric $\bar{g}$ as above is an Einstein metric. (Note that in the preceding problem you said something about the set $\{p \in M : N(p) = 0\}$ where the metric $\bar{g}$ may have issues.)

5. Conformal changes of metric.

a. Suppose (M^n, g) is a Riemannian metric and $\tilde{g} = e^{\varphi} g$. Show that

$$R(\tilde{g}) = e^{-\varphi}\left(R(g) - (n-1)\Delta_g\varphi - \frac{1}{4}(n-1)(n-2)|\nabla\varphi|_g^2\right).$$

b. In case $n \geq 3$, if we write $e^{\varphi} = u^{\frac{4}{n-2}}$ for $u > 0$, then

$$R(\tilde{g}) = u^{-\frac{n+2}{n-2}}\left(R(g)u - \frac{4(n-1)}{(n-2)}\Delta_g u\right).$$

c. Suppose M is compact with empty boundary. Let $c(n) = \frac{n-2}{4(n-1)}$. Let $L_g u = \Delta_g u - c(n)R(g)u$, the *conformal Laplacian*. Show that the total scalar curvature of $\tilde{g} = u^{\frac{4}{n-2}} g$ is given by

$$\int_M R(\tilde{g})\, dv_{\tilde{g}} = c(n)^{-1}\int_M \left(|\nabla u|_g^2 + c(n)R(g)u^2\right) dv_g.$$

HINT: Show that $dv_{\tilde{g}} = u^{\frac{2n}{n-2}}\, dv_g$.

6. SOME ASYMPTOTIC EXPANSIONS. Suppose $(\mathbb{R}^3 \setminus \overline{B_{r_0}(0)}, g)$ is harmonically flat: $g = u^4 g_{\mathbb{E}}$, $R(g) = 0$, i.e. $\Delta_{g_{\mathbb{E}}} u = 0$, with $u(x) \to 1$ as $|x| \to +\infty$. We saw the expansion $u(x) = 1 + \frac{A}{|x|} + \frac{\beta_i x^i}{|x|^3} + O(|x|^{-3})$ via spherical harmonics.

a. Let $x = y + c$, for $c \in \mathbb{R}^3$. For $|y + c| > r_0$, find the asymptotic expansion of u as a function of y. Show for $A \neq 0$ that there is a unique choice of $c \in \mathbb{R}^3$ for which $\tilde{u}(y) := u(y+c) = 1 + \frac{A}{|y|} + O(|y|^{-3})$.

b. Compute $\lim\limits_{r\to+\infty} \int\limits_{|x|=r} x^k \sum\limits_{i,j=1}^{3} (g_{ij,i} - g_{ii,j})\, \nu_e^j\, d\sigma_e$ where $\nu_e^j = \frac{x^j}{r}$. (Warning: this gives the center of mass, but the flux integral isn't the right form for more general asymptotically flat metrics.)

7. CONSTRAINTS MAP IN HARMONIC ASYMPTOTICS. Define the operator $(\widetilde{\mathcal{L}}_g(X))_{ij} = X_{i;j} + X_{j;i} - X^k_{\ ;k} g_{ij}$. If γ is a metric on M^3, let $g = u^4\gamma$ and $\pi_{ij} = u^2(\widetilde{\mathcal{L}}_\gamma(X))_{ij}$ for $u > 0$.

a. Compute the constraints map $\Phi(g, \pi) = (R(g) - |\pi|_g^2 + \frac{1}{2}(\mathrm{tr}_g\pi)^2, \mathrm{div}_g\pi)$, and in case $\gamma = g_{\mathbb{E}}$, show that the vacuum constraints $\Phi(g, \pi) = \mathbf{0}$ can be written, in a Cartesian coordinate system for the background $g_{\mathbb{E}}$, as follows (subscripts for the flat metric omitted):

$$8\Delta u = u\big(-|\widetilde{\mathcal{L}}X|^2 + \tfrac{1}{2}(\mathrm{tr}(\widetilde{\mathcal{L}}X))^2\big)$$
$$\Delta X^i + 4u^{-1}u_{,j}(\widetilde{\mathcal{L}}X)_i^j - 2u^{-1}u_{,i}\mathrm{tr}(\widetilde{\mathcal{L}}X) = 0$$

b. If the above equations in part a. hold on an asymptotic end of an AF manifold (M, g), one can show that u and X have partial expansions $u(x) = 1 + \frac{A}{|x|} + O(|x|^{-2})$, $X^i(x) = \frac{B^i}{|x|} + O(|x|^{-2})$, along with fall off for derivatives. Show that $\pi_{ij} = -\frac{B^i x^j + B^j x^i}{|x|^3} + \sum\limits_k \frac{B^k x^k}{|x|^3}\delta_{ij} + O(|x|^{-3})$, and that $P^i = -\frac{B^i}{2}$ is the ADM linear momentum.

8. Assume that h is a (smooth) transverse-traceless tensor at the Euclidean metric on $\mathbb{R}^3$. Let's use Cartesian coordinates x, so that covariant derivative components are computed via partial derivatives (the Christoffel symbols vanish). So $0 = \mathrm{tr}_{g_{\mathbb{E}}} h = \sum\limits_{i=1}^{3} h_{ii}$, and $0 = (\mathrm{div}_{g_{\mathbb{E}}} h)_j = \sum\limits_{i=1}^{3} h_{ij,j}$. Now,

assume that h has compact support. Let $\gamma_\epsilon = g_{\mathbb{E}} + \epsilon h$, and for $|\epsilon|$ sufficiently small, let $u_\epsilon > 0$ be the associated conformal factor so that with $g_\epsilon = u_\epsilon^4 \gamma_\epsilon$, $R(g_\epsilon) = 0$, and u_ϵ tends to 1 at infinity. Near infinity each u_ϵ is harmonic, and as such has an asymptotic expansion $u_\epsilon = 1 + \frac{m(\epsilon)}{2|x|} + O(|x|^{-2})$.

a. Show that $16\pi m(\epsilon) = -\int\limits_{\mathbb{R}^3} R(\gamma_\epsilon) u_\epsilon \, dv_{g_\epsilon}$.

b. Show that $m'(0) = 0$ and that $16\pi m''(0) = \frac{1}{2} \int\limits_{\mathbb{R}^3} |\nabla_{g_{\mathbb{E}}} h|^2 \, dv_e$.

9. Rigidity computations for PET. Suppose (M, g) is asymptotically flat with one end, with $R(g) \geq 0$, $R(g) \in L^1(M)$, and with ADM mass $m = 0$.

a. One can show that there is a conformal factor $u > 0$, $u \to 1$ near infinity, with $u(x) = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{-n+1})$, so that $R(u^{\frac{4}{n-2}} g) = 0$. Argue that $A \leq 0$, and that the mass of $u^{\frac{4}{n-2}} g$ is $m + 2A \leq m$. Show that if $R(g)$ does not vanish identically, then $A < 0$.

b. Suppose $m = 0$. By part a., we have $R(g) = 0$. We want to show the Ricci curvature vanishes. Let $0 \leq \zeta_\theta \leq 1$ be a compactly-supported bump function which is identically 1 on the compact core of M, out to $|x| \leq \theta$, and $\zeta_\theta = 0$ outside $|x| \geq 2\theta$ (or $|x| \geq \theta + 1$, say). Let $h_\theta = \zeta_\theta \mathrm{Ric}(g)$. Let $\gamma_\epsilon = g + \epsilon h_\theta$. Let $u_\epsilon > 0$ be so that $R(u_\epsilon^{\frac{4}{n-2}} \gamma_\epsilon) = 0$, with $u_\epsilon \to 1$ at infinity. (Such a function exists for small ϵ, because $\Delta_g - \frac{n-2}{4(n-1)} R(\gamma_\epsilon)$ is a small perturbation of Δ_g, which is invertible in suitable weighted function spaces.) Let $g_\epsilon = u_\epsilon^{\frac{4}{n-2}} \gamma_\epsilon$, and let $m(\epsilon)$ be the ADM mass of g_ϵ. Use the idea of #8a., and argue that $m'(0) = 0$. Then compute $m'(0)$ using the linearization of scalar curvature operator (#4c.) $DR_g(h) = -\Delta_g(\mathrm{tr}_g h) + \mathrm{div}_g \mathrm{div}_g h - \langle h, \mathrm{Ric}(g) \rangle_g$.

c. From part b. we have an complete manifold (M, g) with vanishing Ricci curvature, which is also asymptotically flat. Use the Bishop-Gromov volume comparison to argue that (M, g) must be isometric to $(\mathbb{R}^n, g_{\mathbb{E}^n})$.

10. This problem refers to Proposition 3.2 in the Corvino-Pollack article. The proof as written has a gap. It was first written for $R(g) = 0$ or small, but the point of the way it is stated is to allow more general $R(g) \geq 0$. Locate the error. Then fix it! To do so, note that *what is small* is $R(g_\theta) - \psi_\theta R(g)$ for large θ. Modify the desired scalar curvature of $R(u^4 g_\theta)$ to make the resulting PDE to solve for u to be much nicer, and then complete the proof.

Appendix: Problems on Euclidean Harmonic Functions.

1. a. Verify that the following distributional equations hold: $\Delta(\frac{1}{2\pi} \log |x|) = \delta_0$ in dimension $n = 2$, while $\Delta(\frac{1}{(2-n)n\omega_n} |x|^{2-n}) = \delta_0$ in dimensions $n > 2$. Here δ_0 is the Dirac delta distribution at the origin.

b. Suppose $f \in C_c^2(\mathbb{R}^n)$, $n > 2$. Suppose $\mathrm{spt}(f) \subset \{x : |x| \leq K\}$. Then if we let $u(x) = \frac{1}{(2-n)n\omega_n} \int\limits_{\mathbb{R}^n} |x - y|^{2-n} f(y) \, dy$, then $\Delta u = f$ by the above. Moreover, show that u has an expansion of the form $u(x) = \frac{A}{|x|^{n-2}} + \frac{B_i x^i}{|x|^n} + O(|x|^{-n})$. Express the constants A and B_i in terms of integrals involving f.

2. a. Show that if u is harmonic with an isolated singularity at $x = 0$, then the singularity is in fact removable if $\lim_{x \to 0} |x|^{n-2} u(x) = 0$ in case $n > 2$, and in case $n = 2$, if $\lim_{x \to 0} \frac{u(x)}{\log |x|} = 0$.

b. If $K[u]$ is the Kelvin transform of u, find $\Delta(K[u])$ in terms of Δu. Conclude that $K[u]$ is harmonic if and only if u is harmonic. Recall $K[u](x) = |x|^{2-n} u(x^*)$, $x^* = |x|^{-2} x$.

c. Prove that if $n > 2$ and u is harmonic near infinity. Prove that u is harmonic at infinity if and only if $\lim_{|x| \to +\infty} u(x) = 0$.

3. If v is harmonic at infinity and $n > 2$, v admits an expansion at infinity in terms of spherical harmonics. We derived the first two terms which give $v(x) = \frac{a_0}{|x|^{n-2}} + \frac{a_i x^i}{|x|^n} + O(|x|^n)$. Derive the next order term, in case $n = 3$.

4. Let $(\mathbb{S}^n, g_0)$ be the standard unit round sphere, $\mathbb{S}^n$ embedded in $\mathbb{R}^{n+1}$ as $\{|x| = 1\}$. It is a fact that the lowest positive eigenvalue λ_1 for Δ_{g_0} corresponds to the eigenfunctions x^i (Euclidean coordinates) restricted to the sphere. Compute $\lambda_1 = n$ by using $\Delta_{g_0}(x^i) = -\lambda_1 x^i$. Multiply by x^i, integrate by parts, and use the fact that $\nabla_{g_0} x^i$ is the tangential component of $\nabla x^i = e_i = \frac{\partial}{\partial x^i}$.

5. Recall Bôcher's Theorem: if $u > 0$ is harmonic in a punctured ball $B \setminus \{0\}$, there exist v harmonic in B and $b \geq 0$ so that $u(x) = \begin{cases} b \log(\frac{1}{|x|}) + v(x) , & n = 2 \\ b|x|^{2-n} + v(x) , & n > 2. \end{cases}$

a. Show that b and v are uniquely determined.

b. $\Omega \subset \mathbb{R}^n$ is an open set, $n > 2$. If u is harmonic in $\Omega \setminus \{a\}$ ($a \in \Omega$), so that $u > 0$ in a deleted neighborhood of a, show there is a number $b \geq 0$ and a function v harmonic on all of Ω so that on $\Omega \setminus \{a\}$, $u(x) = b|x - a|^{2-n} + v(x)$.

c. $n > 2$. If u is harmonic on $B \setminus \{0\}$, and $\liminf_{x \to 0} |x|^{n-2} u(x) > -\infty$, there exists v harmonic in B, $b \in \mathbb{R}$ so that $u(x) = b|x|^{2-n} + v(x)$ on $B \setminus \{0\}$.

d. What can you say about a positive harmonic function on $\mathbb{R}^n \setminus \{0, a\}$, $a \neq 0$?