Lecture II

An introduction to (Lagrangian) mean curvature flow with high codimension

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Mean curvature flow (MCF)

- Let $\Sigma^n$ be a smooth submanifold in a Riemannian manifold $N$. If there is a family of smooth immersions $X_t : \Sigma^n \to N$ satisfying

$$\left\{ \begin{align*}
(\frac{\partial X_t(x)}{\partial t})^\perp &= H(X_t(x)) \\
X_0 &= Id
\end{align*} \right.$$ 

then $\Sigma_t$ is called a MCF of $\Sigma$.

- $H = g^{ij}(\nabla \frac{\partial X}{\partial x_i} \frac{\partial X}{\partial x_j})^\perp$, where $g_{ij} = \langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \rangle$.

- A nonlinear weakly parabolic quasi-linear system, when coupling with a diffeomorphism of $\Sigma$, the flow can be made into a normal deformation.
\[ \frac{dA_t}{dt} \bigg|_{t=0} = -\int_{\Sigma} \langle H, V \rangle \, d\Sigma \]

MC\, \bar{H} \, decrease \, the \, area \, most \, rapidly

\[ (\frac{\partial X_t}{\partial t})^+ = H \Rightarrow \text{stationary pte are minimal} \]

- Also along the flow, geometry improves
  - ex: isoperimetric ratio

- Embedded closed plane curves shrink to round pts (Gage-Hamilton-Grayson)

- Embedded convex closed hyper surface in \( \mathbb{R}^n \) shrink to round points (Huisken)
- **Short time existence** for any smooth compact initial data.
- Singularities may occur at larger time. The flow forms singularities when the second fundamental form of the submanifold blows up.
- Use parabolic blow up to understand the profile of isolated singularity.
- Assume $N = \mathbb{R}^m$ now for simplicity. A parabolic dilation of scale $\lambda > 0$ at $(x_0, t_0)$ is defined to be

$$D_\lambda : (x, t) \mapsto (\lambda(x - x_0), \lambda^2(t - t_0)).$$

- Denote $s = \lambda^2(t - t_0)$ and $\Sigma^s_\lambda = D_\lambda(\Sigma_{t_0} + \frac{s}{\lambda^2})$. Then $\Sigma^s_\lambda$ also satisfies MCF and $t \in [0, t_0) \Rightarrow s \in \left(-\lambda^2t_0, 0\right].$
Avoid principle

Let \( \{M^1_t\}, \{M^2_t\}, t \in [a,b] \) be two family of compact embedded hypersurfaces evolving by the mean curvature flow. Suppose \( M^1_a \cap M^2_a = \emptyset \). Then

\[
\frac{d}{dt} \text{dist}(M^1_t, M^2_t) \geq 0
\]

at points of differentiability. In particular, their distance is increasing.

(Idea): Choose \( x_1, x_2 \) realizing distance. Compute \( \frac{d}{dt} \text{dist}(x_1, x_2) \).

If < 0, lead to contradiction.

- For sphere of radius \( r_0 \) in \( \mathbb{R}^{n+1} \), under MCF, solv is sphere of radius \( \sqrt{r_0^2 - 2nt} \)

\[
H = -\frac{n}{\gamma(t)} r_0 \quad F_0 = \text{unit sphere}
\]

- Dumb bell \( \Rightarrow \) Grayson's result not hold for \( n \geq 2 \)

 Avoid principle is not true for higher codim MCF.
Note: MCF of any closed submanifold in $\mathbb{R}^n$ must develop finite time singularity: \[
\frac{d}{dt} |x|^2 = \Delta_t |x|^2 - 2n \Rightarrow \frac{d}{dt} (|x|^2 + 2nt) = \Delta_t (|x|^2 + 2nt)
\]

If $|x_0|^2 \leq R_0^2$, max principle $\Rightarrow |x(t)|^2 + 2nt \leq R_0^2$

Huisken used normalized eq. to analyze the case (hypersurface).

In general manifold, the situation could be different.

Difficulties for MCF in high codim

1. The flow is a non-linear system, cannot reduce to one scalar equation as in codim 1
   (Codim 1, good maximum principle was avoid principle)
② 2nd. fundamental form. symmetric two tensors valued in non-trivial normal bundle
no natural convex condition
What still continue to hold?
① Brakke’s regularity Thm
② Hamilton’s maximum principle for tensors
③ Huisken’s monotonicity formula
④ White’s regularity Thm
To study the flow, one need to look into the evolution eqn of $H$. If $1A^2$, 

In higher codimension, it is much more complicated. 

For a map $f_0: M_1 \rightarrow M_2$, Wang studied MCF of its graph $\Sigma f_0 \subset (M_1, C_1) \times (M_2, C_2)$. Using singular decomposition of the differential (Liouville eigenvalues of $\sqrt{(df)^T df}$) and suitable basis, the evolution eq can be simplified. Evolution eq of $x\Omega = \frac{1}{\sqrt{1 + x^2}}$ is particularly important. (A generalization of $\frac{1}{\sqrt{1 + x^2}}$ for hypersurface $M_2 = R$)
Various curvature conditions on $\Sigma_1, M_2$, and singular value restriction on $g_0$, $h$, and with his collaborators can prove:

1. flow remains graph
2. the flow exists smoothly for all time \( (\text{White's Thm}) \)
3. $f_t$ converges to a constant map as $t$ tends to $\infty$

**Theorem A.** Let $(\Sigma_1, g)$ and $(\Sigma_2, h)$ be Riemannian manifolds of constant curvature $k_1$ and $k_2$ respectively and $f$ be a smooth map from $\Sigma_1$ to $\Sigma_2$. Suppose $k_1 \geq |k_2|$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, the mean curvature flow of the graph of $f$ remains a graph and exists for all time.

\[ \prod (1 + \lambda^*_i) < 2 \] (Wang 2002)

The mean curvature flow for graphs appears to favor positively curved domain manifold. The convergence theorem is the following.

**Theorem B.** Let $(\Sigma_1, g)$ and $(\Sigma_2, h)$ be Riemannian manifolds of constant curvature $k_1$ and $k_2$ respectively and $f$ be a smooth map from $\Sigma_1$ to $\Sigma_2$. Suppose $k_1 \geq |k_2|$ and $k_1 + k_2 > 0$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, then the mean curvature flow of the graph of $f$ converges to the graph of a constant map at infinity.
THEOREM 1.1 Let $\Sigma_1$ and $\Sigma_2$ be compact Riemannian manifolds of constant curvature $k_1$ and $k_2$, respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 > 0$, and $\dim(\Sigma_1) \geq 2$. If $f$ is a smooth, area-decreasing map from $\Sigma_1$ to $\Sigma_2$, the mean curvature flow of the graph of $f$ remains the graph of an area-decreasing map, exists for all time, and converges smoothly to the graph of a constant map.

COROLLARY 1.2 Any area-decreasing map from $S^n$ to $S^m$ with $n \geq 2$ is homotopically trivial.

(Lee, 2011)

Theorem 1. Let $(N_1, g)$ and $(N_2, h)$ be two compact Riemannian manifolds, and let $f$ be a smooth map from $N_1$ to $N_2$. Assume that $K_{N_1} \geq k_1$ and $K_{N_2} \leq k_2$ for two constants $k_1$ and $k_2$, where $K_{N_1}$ and $K_{N_2}$ are the sectional curvature of $N_1$ and $N_2$, respectively. Suppose either $k_1 \geq 0, k_2 \leq 0$, or $k_1 \geq k_2 > 0$. Then the following results hold:

(i) If \( \frac{\det((g+f^*h)_{ij})}{\det(g_{ij})} < 4 \), then the mean curvature flow of the graph of $f$ remains the graph of a map and exists for all time.
(ii) Furthermore, if $k_1 > 0$, then the mean curvature flow converges smoothly to the graph of a constant map.
Theorem 2. Assume the same conditions as in Theorem \[1\]. Then the following results hold:

(i) If \( f \) is a smooth area-decreasing map from \( N_1 \) to \( N_2 \), then the mean curvature flow of the graph of \( f \) remains the graph of an area-decreasing map and exists for all time.

(ii) Furthermore, if \( k_1 > 0 \), then the mean curvature flow converges smoothly to the graph of a constant map.

(Savare-Hajilouj and Smoczyk, 2014)

Theorem A. Let \( M \) and \( N \) be two compact Riemannian manifolds. Assume that \( m = \dim M \geq 2 \) and that there exists a positive constant \( \sigma \) such that the sectional curvatures \( \sec_M \) of \( M \) and \( \sec_N \) of \( N \) and the Ricci curvature \( \text{Ric}_M \) of \( M \) satisfy

\[
\sec_M > -\sigma, \quad \text{Ric}_M \geq (m-1)\sigma \geq (m-1)\sec_N.
\]

If \( f : M \to N \) is a strictly area decreasing smooth map, then the mean curvature flow of the graph of \( f \) remains the graph of a strictly area decreasing map and exists for all time. Moreover, under the mean curvature flow the area decreasing map converges to a constant map.
Theorem 4.1 Let $\Sigma^1$ and $\Sigma^2$ be two compact closed Riemann surfaces with metrics of the same constant curvature $c$. Let $\omega_1$ and $\omega_2$ be the volume forms of $\Sigma^1$ and $\Sigma^2$, respectively. Consider a map $f : \Sigma^1 \to \Sigma^2$ that satisfies $f^*\omega_2 = \omega_1$, i.e. $f$ is an area-preserving map or a symplectomorphism. Denote by $\Sigma_t$ the mean curvature flow of the graph of $f$ in $M = \Sigma^1 \times \Sigma^2$. We have

1. $\Sigma_t$ exists smoothly for all $t > 0$ and converges smoothly to $\Sigma_\infty$ as $t \to \infty$.

2. Each $\Sigma_t$ is the graph of a symplectomorphism $f_t : \Sigma^1 \to \Sigma^2$ and $f_t$ converges smoothly to a symplectomorphism $f_\infty : \Sigma^1 \to \Sigma^2$ as $t \to \infty$.

Moreover,

$$f_\infty \text{ is } \begin{cases} 
\text{an isometry} & \text{if } c > 0 \\
\text{a linear map} & \text{if } c = 0 \\
\text{a harmonic diffeormorphism} & \text{if } c < 0
\end{cases}$$

Since any diffeomorphism is isotopic to an area preserving diffeomorphism, this gives a new proof of Smale’s theorem [29] that $O(3)$ is the deformation retract of the diffeomorphism group of $S^2$. For a positive genus Riemann surface, this implies the identity component of the diffeomorphism group is contractible.
Theorem 4 [24] There exists an explicitly computable constant $\Lambda > 1$ depending only on $n$, such that any symplectomorphism $f: \mathbb{C}P^n \to \mathbb{C}P^n$ with

$$\frac{1}{\Lambda} g \leq f^* g \leq \Lambda g$$

is symplectically isotopic to a biholomorphic isometry of $\mathbb{C}P^n$ through the mean curvature flow.

A theorem of Gromov [15] shows that, when $n = 2$, the statement holds true without any pinching condition by the method of pseudoholomorphic curves. Our theorem is not strong enough to give an analytic proof of Gromov’s theorem for $n = 2$. However, for $n \geq 3$, this seems to be the first known result.

Theorem A. Let $\Sigma$ be a Lagrangian submanifold in $T^{2n}$. Suppose $\Sigma$ is the graph of $f: T^n \to T^n$ and the potential function $u$ of $f$ is convex. Then the mean curvature flow of $\Sigma$ exists for all time and converges smoothly to a flat Lagrangian submanifold.

$$\frac{du}{dt} = \frac{1}{\sqrt{-1}} \ln \frac{\det(I + \sqrt{-1} D^2 u)}{\sqrt{\det(I + (D^2 u)^2)}}.$$ from $\frac{d}{dt} U_i = g^{sk} U_{ijk}$
Consider the fully nonlinear parabolic equation for a function $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$

\[
\begin{cases}
\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \arctan \lambda_i = 0 \\
u(x, 0) = u_0(x)
\end{cases}
\]

**Theorem 1.1** Suppose that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a function with $L^\infty$ Hessian satisfying

\[-(1 - \delta)I_n \leq \text{ess inf } D^2u_0 \leq \text{ess sup } D^2u_0 \leq (1 - \delta)I_n \quad \text{(1.3)}\]

for any $\delta \in (0, 1)$. Then (1.1) has a longtime smooth solution $u(x, t)$ for all $t > 0$ with initial condition $u_0$ such that the following estimates hold:

1. $-(1 - \delta)I_n \leq D^2u \leq (1 - \delta)I_n$ for all $t > 0$.
2. $\sup_{x \in \mathbb{R}^n} |D^l u(x, t)|^2 \leq C_{l, \delta}/t^{l-2}$ for all $l \geq 3$, and some $C_{l, \delta}$ depending only on $l, \delta$.
3. $u$ and $Du$ are Hölder continuous in time at $t = 0$ with Hölder exponents 1 and $1/2$ respectively.

If in addition $|Du_0(x)| \to 0$ as $|x| \to \infty$, then $\sup_{x \in \mathbb{R}^n} |Du(x, t)| \to 0$ as $t \to \infty$. In particular, the graph $(x, Du(x, t))$ immediately becomes smooth and converges smoothly on compact sets to the coordinate plane $(x, 0)$ in $\mathbb{R}^{2n}$.

$\Leftrightarrow U$ strictly convex after changing coordinate.
Theorem 1.2 Suppose that \( u_0 : \mathbb{R}^n \to \mathbb{R} \) is a function with \( L^\infty \) Hessian satisfying (1.3) and suppose that

\[
\lim_{\lambda \to \infty} \lambda^{-2} u_0(\lambda x) = U_0(x)
\]  \hspace{1cm} (1.4)

for some \( U_0(x) \). Let \( u(x, t) \) be the solution to (1.1) with initial data \( u_0(x) \). Then \( \lambda^{-2} u(\lambda x, \lambda^2 t) \) converges to a smooth self-expanding solution \( U(x, t) \) to (1.1) uniformly on compact subsets of \( \mathbb{R}^n \times (0, \infty) \) as \( \lambda \to \infty \), and \( U(x, t) \) converges to \( U_0(x) \) uniformly on compact subsets of \( \mathbb{R}^n \) as \( t \to 0 \). In particular, there is a one-to-one correspondence between self-expanding solutions to (1.1) satisfying (1.3) and Lipschitz functions on \( \mathbb{R}^n \) which are homogeneous of degree 2 and satisfy (1.3).

Chau - Chen - Yuan, 2013

Theorem 1.1 There exists a small positive dimensional constant \( \eta = \eta(n) \) such that if \( u_0 : \mathbb{R}^n \to \mathbb{R} \) is a \( C^{1,1} \) function satisfying

\[-(1 + \eta) I_n \leq D^2 u_0 \leq (1 + \eta) I_n \]  \hspace{1cm} (2)

then (1) has a unique longtime smooth solution \( u(x, t) \) for all \( t > 0 \) with initial condition \( u_0 \) such that the following estimates hold:

(i) \(-\sqrt{3} I_n \leq D^2 u \leq \sqrt{3} I_n \) for all \( t > 0 \).

(ii) \( \sup_{x \in \mathbb{R}^n} |D^l u(x, t)|^2 \leq C_l/t^{l-2} \) for all \( l \geq 3 \), \( t > 0 \) and some \( C_l \) depending only on \( l \).

(iii) \( Du(x, t) \) is uniformly Hölder continuous in time at \( t = 0 \) with Hölder exponent \( 1/2 \).
Monotonicity Formula (Huisken, 1990)

- Back heat kernel at \((x_0, t_0)\)

\[
\rho_{x_0, t_0}(x, t) = \frac{1}{\sqrt{4\pi(t_0 - t)}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}, \quad t < t_0.
\]

- If \(\Sigma_t\) satisfies MCF, then

\[
\frac{d}{dt} \int_{\Sigma_t} \rho_{x_0, t_0} \, d\mu_t = - \int_{\Sigma_t} \rho_{x_0, t_0} |H + \frac{\nabla_x(x - x_0)^\perp}{2(t_0-t)}|^2 \, d\mu_t \leq 0,
\]

\[\Rightarrow \lim_{t \to t_0} \int_{\Sigma_t} \rho_{x_0, t_0} \, d\mu_t \text{ exists, called the density at } (x_0, t_0).
\]

- Have a similar formula for MCF in a Riemannian manifold and the density is also defined (White, 1997).
\[ \int_{\Sigma_t} \rho_{x_0,t_0} d\mu_t \text{ is invariant under dilation. That is,} \]
\[ \int_{\Sigma_t} \rho_{x_0,t_0} d\mu_t = \int_{\Sigma_{s,0}^\lambda} \rho_{0,0} d\mu_s^\lambda. \]

Choose \( \lambda_i \to \infty \). If the limit of the flow of \( \Sigma_{s,0}^{\lambda_i} \) exists, it must satisfy
\[ H(x, s) - \frac{1}{2s} X^\perp_t = 0, \quad s < 0. \]

For MCF in a Riemannian manifold, the blow up limit still lies in \( \mathbb{R}^m \).

\( \Sigma \) is called a (normalized) self-shrinker in \( \mathbb{R}^m \) if its position vector \( X : \Sigma \to \mathbb{R}^m \) satisfies \( H = -\frac{1}{2} X^\perp \). \( \Rightarrow \sqrt{1-tX} \) satisfies MCF.
Soliton Solution to MCF in $\mathbb{R}^m$

Solutions looks the same under MCF, either of the form $\varphi(t)X(x)$ or $X(x) + \psi(t)$ Direct computation shows that

- In case (1), $H = cX^\perp$ and $\varphi(t) = \sqrt{1 + 2ct}$ for a constant $c$. $c < 0$, solutions shrink, self-shrinkers, models for central blowup limits. $c = 0$, solutions do not move, minimal submanifolds. $c > 0$, solutions expand, called self-expanders, possible limit at infinity.

- In case (2), $H = T^\perp$ for a constant vector $T$ and $\psi(t) = tT$, called a translating solution. It is possible limit for max point blowup.

- Find such examples. Try to classify. Determine whether these examples are blowup limits of MCF. Use these examples to understand the singularities, and do surgery.
If $L(x,t)$ satisfies $\text{MCF}_x$, then $\Sigma(x,s) = \sigma \left( \Sigma(x, t_0 + \frac{s}{\sigma^2}) - t_0 \right)$ also satisfies $\text{MCF}_x$. If $\Sigma$ defined in $[0,T)$

\[ \left(\frac{\partial \Sigma}{\partial s}\right)^{-1} = \left(\sigma \frac{\partial \Sigma}{\partial t} - \frac{1}{\sigma^2}\right)^{-1} \]

= \frac{1}{\sigma} \tilde{H}(\Sigma(x,t)) = \tilde{H}(\tilde{\Sigma}(x,s))

1. Type I blow up (central pt blow up)

assume $\Sigma$ has an isolated singularity at $(y_0,T)$

$\sigma \rightarrow \infty$. $\tilde{\Sigma_i}(x,s) = \sigma_i \left( \Sigma(x, t + \frac{s}{\sigma_i^2}) - t_0 \right)$

$\exists$ a subseq. $-\sigma_i^2 T \leq s < 0 \rightarrow \text{ancient sol}$

$\exists \tilde{\Sigma_i}$ have Brakke flow limit $\tilde{\Sigma}^{2\infty}$
Different subseq may have different limits

2. Type II blow up (max pt blow up)

\[ \tilde{\Sigma}(x, S) = \delta_i \left( \sum (x, t_i + \frac{S}{\delta_i^2} - y_i) \right), \quad \delta_i \to \infty \]

\[ y_i = \sum (x_i, t_i), \quad t_i \to T_i \quad \& \quad y_i \to y_0 \]

\[ -t_i \delta_i^2 \leq s \leq (T - t_i) \delta_i^2 \]

\[ s (T - t_i) \delta_i^2 \to 0 \quad \to \text{eternal sol} \]

\[ \exists \text{ a subseq } \Rightarrow \tilde{\Sigma}^i \text{ has a Brakke flow limit } \tilde{\Sigma}^\infty \]

usually take \( \delta_i \sim \max \|A(x, t_i)\| = A(x_i, t_i) \)
A singularity is called type I singularity if

$$\max_{\Sigma_t} |A|^2 \leq \frac{C}{T-t}$$

- $MC^H \text{ in } KE$. Lag condition is preserved for cpt smooth sole Smoczyk 1996

follow from

$$\frac{d}{dt}|\omega|^2 \leq \Delta|\omega|^2 + c|\omega|^2$$

Wang, Chen-Li. Never

No type I singularity for LMCH if graded Lag

- Cpt. + almost Calibrated
- Entire graph

- also for $MC^H$ of symplectic sphere in $KE$, $\omega > 0$
Main reason: all smooth graded Lag shrinkers must be planar from more general form of Huisken's monotonicity formula.

**Theorem**
Let $f_t$ be a smooth family of functions on $2_t$. Then assuming all quantities are finite:

$$\frac{d}{dt}\int_{\Sigma_t} f_t \Phi(x_0, T) d\mathcal{H}^n = \int_{\Sigma_t} (\frac{df_t}{dt} - \Delta f_t) \Phi(x_0, T) d\mathcal{H}^n$$

$$- \int_{\Sigma_t} f_t \mathcal{H}^1 \left(\frac{1}{2(T-t)} \right)^{\frac{1}{2}} \Phi(x_0, T) d\mathcal{H}^n$$

$$\int_{\Sigma_t} \Phi(x_0, T)(x, t) = \frac{\exp \left(-\frac{|x-x_0|^2}{4(T-t)}\right)}{(4\pi(T-t))^{n/2}}.$$  

\[\Theta_t(x_0, l) = \int_{M_t} \Phi(x_0, l)(x, 0) d\mathcal{H}^k.\]

**Theorem 3.1 (White’s Regularity Theorem).** There are $\varepsilon_0 = \varepsilon_0(n, k)$, $C = C(n, k)$ so that if $\partial M_t \cap B_{2R} = \emptyset$ and $M_t$ satisfies $\text{MC} \mathcal{H}$,

$$\Theta_t(x, l) \leq 1 + \varepsilon_0 \quad \text{for all} \quad l \leq R^2, x \in B_{2R}, \text{and} \ t \leq R^2,$$

then the $C^{2, \alpha}$-norm of $M_t$ in $B_R$ is bounded by $C/\sqrt{t}$ for all $t \leq R^2$. 

Being graded Lag, we have an extra function $Q$ preserved under $\text{LMC}_t$:

$$\frac{d}{dt} Q_t = \Delta Q_t \Rightarrow \frac{d}{dt} \Theta_t^2 = \Delta \Theta_t^2 - 2|H|^2$$

1. smooth graded Lag shrunken. $H = -\frac{1}{2} X^\perp$ (assuming $Q$ bded)

$L_t = \sqrt{-t} L$ satisfies $\text{LMC}_t$ &

$$H_t = \frac{1}{\sqrt{-t}} H = -\frac{1}{2} \frac{1}{\sqrt{-t}} X^\perp = -\frac{1}{2} \frac{1}{\sqrt{-t}} \frac{1}{\sqrt{-t}} X_t^\perp = -\frac{1}{2} \frac{X_t^\perp}{-t}.$$

Define $Q(t) = \int_{L_t} \Theta^2 \Phi(0,0) dA^n$ scaling invariant

$$\Rightarrow 0 = \frac{d}{dt} Q = \int_{L_t} (\frac{d}{dt} - \Delta) \Theta^2 \Phi(0,0) dA^n - \int_{L_t} \Theta^2 |H| - \frac{X_t^\perp}{2t} \int_{L_t} ^2 \Phi(0,0) dA^n$$

$$= -2 \int_{L_t} |H|^2 \Phi(0,0) dA^n \Rightarrow H_t = 0 \text{ minimal cone}$$

I smooth : plane