Lecture II

# An introduction to (Lagrangian) mean curvature flow with high codimension

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## Mean curvature flow (MCF)

- Let $\Sigma^n$ be a smooth submanifold in a Riemannian manifold $N$. If there is a family of smooth immersions $X_t : \Sigma^n \to N$ satisfying

$$\begin{cases} \left(\dfrac{\partial X_t(x)}{\partial t}\right)^{\perp} = H(X_t(x)) \\ X_0 = Id \end{cases}$$

  then $\Sigma_t$ is called a MCF of $\Sigma$.
- $H = g^{ij}(\nabla_{\frac{\partial X}{\partial x_i}} \frac{\partial X}{\partial x_j})^{\perp}$, where $g_{ij} = \langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \rangle$.
- A nonlinear weakly parabolic quasi-linear system, when coupling with a diffeomorphism of $\Sigma$, the flow can be made into a normal deformation.

$$\frac{dA_t}{dt}\bigg|_{t=0} = -\int_\Sigma \langle \vec{H}, V \rangle \, d\mathrm{Vol}_\Sigma$$

MCF decreases the area most rapidly

- $\left(\frac{\partial X_t}{\partial t}\right)^\perp = H \Rightarrow$ stationary pts are minimal
- Also along the flow, geometry improves

  ex: isoperimetric ratio
- Embedded closed plane curves shrink to round pts

  (Gage - Hamilton - Grayson)
- Embedded convex closed hypersurface in $\mathbb{R}^n$ shrink to round points (Huisken)

true for some complete non compact case

- Short time existence for any smooth compact initial data.
- Singularities may occur at larger time. The flow forms singularities when the second fundamental form of the submanifold blows up.

unique Lip

- Use parabolic blow up to understand the profile of isolated singularity.
- Assume $N = \mathbb{R}^m$ now for simplicity. A parabolic dilation of scale $\lambda > 0$ at $(x_0, t_0)$ is defined to be

$$D_\lambda : (x,t) \mapsto (\lambda(x - x_0), \lambda^2(t - t_0)).$$

- Denote $s = \lambda^2(t - t_0)$ and $\Sigma_s^\lambda = D_\lambda(\Sigma_{t_0 + \frac{s}{\lambda^2}})$. Then $\Sigma_s^\lambda$ also satisfies MCF and $t \in [0, t_0) \Rightarrow s \in \left(-\lambda^2 t_0, 0\right]$.

## avoid principle

[redacted]. *Let $\{M_t^1\}, \{M_t^2\}, t \in [a,b]$ be two family of compact embedded hypersurfaces evolving by the mean curvature flow. Suppose $M_a^1 \cap M_a^2 = \phi$. Then*

$$\frac{d}{dt}\,\mathrm{dist}(M_t^1, M_t^2) \geq 0$$

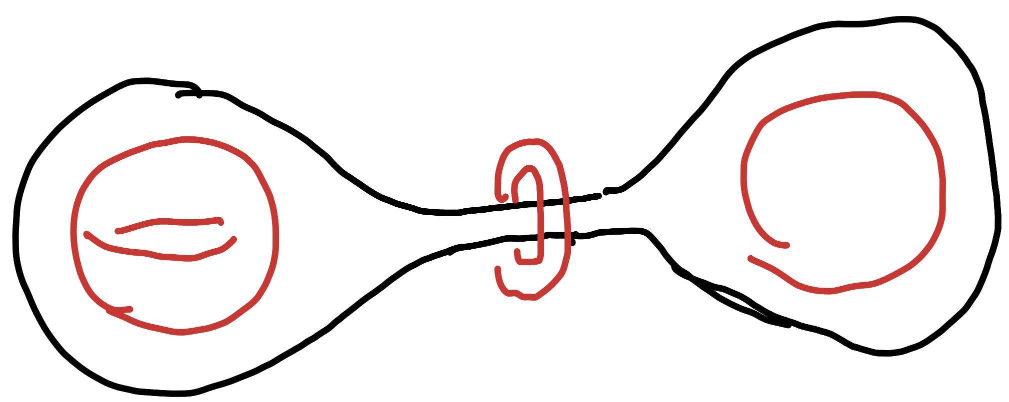
*at points of differentiability. In particular, their distance is increasing.*

(Idea): choose $x_1, x_2$ realizing distance. Compute $\frac{d}{dt}\,\mathrm{dist}(x_1, x_2)$.
If $< 0$. lead to contradiction

- For sphere of radius $r_0$ in $\mathbb{R}^{n+1}$ ($\mathbb{R}^N$), under MCF. sol is sphere of radius $\sqrt{r_0^2 - 2nt}$

$H = -\frac{n}{r(t)} F_0$ $\quad$ $F_0$ : unit sphere

- Dumb bell



$\Rightarrow$ Grayson's result not hold for $n \geq 2$

- avoid principle is not true for higher codim MCF.

Note: MCF of any closed submfd in $\mathbb{R}^N$ must develop finite time singularity: $\frac{d}{dt}|X|^2 = \Delta_t |X|^2 - 2n \Rightarrow \frac{d}{dt}(\|X\|^2 + 2nt) = \Delta_t(\|X\|^2 + 2nt)$

If $\|X_0\|^2 \le r_0^2$, max principle $\Rightarrow \|X_t\|^2 + 2nt \le r_0^2$

Huisken uses normalized eq. to analyze the case (hypersurface)

In general manifold, the situation could be different.

Difficulties for MCF in high codim

① The flow is a nonlinear system, cannot reduce to one scalar equation as in codim 1

( codim 1. good maximum principle vs avoid principle )

② 2nd. foundamental form . symmetric two tensors valued in

nontrivial normal bundle

⇝ no natural convex condition

What still continue to hold ?

① Brakke's regularity Thm

② Hamilton's maximum principle for tensors

③ Huisken's monotonicity formula

④ White's regularity Thm

To study the flow, one need to look into the evolution eqn of $g$. $H$. $|A|^2$. ---

In higher codimension, it is much more complicated.

For a map $f_0 : M_1 \longmapsto M_2$, Wang studied MCF of its graph $\Sigma_{f_0} \subset (M_1, g_1) \times (M_2, g_2)$. Using singular decomposition of the differential ($\{\lambda_i\}$ eigenvalues of $\sqrt{(df^T df)}$) and suitable basis. the evolution eq can be simplied. Evolution eq of $*\Omega = \dfrac{1}{\sqrt{\prod(1+\lambda_i^2)}}$ is particularly important. (a generalization of $\dfrac{1}{\sqrt{1+|\nabla f|^2}}$ for hypersurface $M_2 = \mathbb{R}$)

- various curvature conditions on $M_1, M_2$. and singular value restriction on $g_0$, he, and with his collaborators can prove

  ⓐ flow remains graph

  ⓑ the flow exists smoothly for all time (White's Thm)

  ⓒ $f_t$ converges to a constant map as $t$ tends to $\infty$

**Theorem A.** *Let $(\Sigma_1, g)$ and $(\Sigma_2, h)$ be Riemannian manifolds of constant curvature $k_1$ and $k_2$ respectively and $f$ be a smooth map from $\Sigma_1$ to $\Sigma_2$. Suppose $k_1 \geq |k_2|$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, the mean curvature flow of the graph of $f$ remains a graph and exists for all time.*

(Wang 2002)

$\prod(1+\lambda_i^2) < 2$

The mean curvature flow for graphs appears to favor positively curved domain manifold. The convergence theorem is the following.

**Theorem B.** *Let $(\Sigma_1, g)$ and $(\Sigma_2, h)$ be Riemannian manifolds of constant curvature $k_1$ and $k_2$ respectively and $f$ be a smooth map from $\Sigma_1$ to $\Sigma_2$. Suppose $k_1 \geq |k_2|$ and $k_1 + k_2 > 0$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, then the mean curvature flow of the graph of $f$ converges to the graph of a constant map at infinity.*

(Wang - Tsui 2004)

THEOREM 1.1 *Let $\Sigma_1$ and $\Sigma_2$ be compact Riemannian manifolds of constant curvature $k_1$ and $k_2$, respectively. Suppose $k_1 \geq |k_2|$, $k_1 + k_2 > 0$, and $\dim(\Sigma_1) \geq 2$. If $f$ is a smooth, area-decreasing map from $\Sigma_1$ to $\Sigma_2$, the mean curvature flow of the graph of $f$ remains the graph of an area-decreasing map, exists for all time, and converges smoothly to the graph of a constant map.*

$\lambda_i \lambda_j < 1$

COROLLARY 1.2 *Any area-decreasing map from $\mathbb{S}^n$ to $\mathbb{S}^m$ with $n \geq 2$ is homotopically trivial.*

(L - Lee, 2011)

**Theorem 1.** *Let $(N_1, g)$ and $(N_2, h)$ be two compact Riemannian manifolds, and let $f$ be a smooth map from $N_1$ to $N_2$. Assume that $K_{N_1} \geq k_1$ and $K_{N_2} \leq k_2$ for two constants $k_1$ and $k_2$, where $K_{N_1}$ and $K_{N_2}$ are the sectional curvature of $N_1$ and $N_2$, respectively. Suppose either $k_1 \geq 0, k_2 \leq 0$, or $k_1 \geq k_2 > 0$. Then the following results hold:*

(i) *If $\frac{\det((g+f^*h)_{ij})}{\det(g_{ij})} < 4$, then the mean curvature flow of the graph of $f$ remains the graph of a map and exists for all time.*

(ii) *Furthermore, if $k_1 > 0$, then the mean curvature flow converges smoothly to the graph of a constant map.*

**Theorem 2.** *Assume the same conditions as in Theorem 1. Then the following results hold:*

(i) *If $f$ is a smooth area-decreasing map from $N_1$ to $N_2$, then the mean curvature flow of the graph of $f$ remains the graph of an area-decreasing map and exists for all time.*

(ii) *Furthermore, if $k_1 > 0$, then the mean curvature flow converges smoothly to the graph of a constant map.*

(Savas-Halilaj. and Smoczyk, 2014)

**Theorem A.** *Let $M$ and $N$ be two compact Riemannian manifolds. Assume that $m = \dim M \geqslant 2$ and that there exists a positive constant $\sigma$ such that the sectional curvatures $\sec_M$ of $M$ and $\sec_N$ of $N$ and the Ricci curvature $\mathrm{Ric}_M$ of $M$ satisfy*

$$\sec_M > -\sigma, \qquad \mathrm{Ric}_M \geqslant (m-1)\sigma \geqslant (m-1)\sec_N .$$

*If $f : M \to N$ is a strictly area decreasing smooth map, then the mean curvature flow of the graph of $f$ remains the graph of a strictly area decreasing map and exists for all time. Moreover, under the mean curvature flow the area decreasing map converges to a constant map.*

Wang 2001, 2005

**Theorem 4.1** *Let $\Sigma^1$ and $\Sigma^2$ be two compact closed Riemann surfaces with metrics of the same constant curvature $c$. Let $\omega_1$ and $\omega_2$ be the volume forms of $\Sigma^1$ and $\Sigma^2$, respectively. Consider a map $f : \Sigma^1 \to \Sigma^2$ that satisfies $f^*\omega_2 = \omega_1$, i.e. $f$ is an area-preserving map or a symplectomorphism. Denote by $\Sigma_t$ the mean curvature flow of the graph of $f$ in $M = \Sigma^1 \times \Sigma^2$. We have*

1. *$\Sigma_t$ exists smoothly for all $t > 0$ and converges smoothly to $\Sigma_\infty$ as $t \to \infty$.*
2. *Each $\Sigma_t$ is the graph of a symplectomorphism $f_t : \Sigma^1 \to \Sigma^2$ and $f_t$ converges smoothly to a symplectomorphism $f_\infty : \Sigma^1 \to \Sigma^2$ as $t \to \infty$.*

*Moreover,*

min Lag

$$f_\infty \text{ is } \begin{cases} \text{an isometry} & \text{if } c > 0 \\ \text{a linear map} & \text{if } c = 0 \\ \text{a harmonic diffeormophism} & \text{if } c < 0 \end{cases}$$

Since any diffeomorphism is isotopic to an area preserving diffeomorphism, this gives a new proof of Smale's theorem [29] that O(3) is the deformation retract of the diffeomorphism group of $S^2$. For a positive genus Riemann surface, this implies the identity component of the diffeomorphism group is contractible.

(Medos– Wang, 2011)

**Theorem 4** [24] *There exists an explicitly computable constant $\Lambda > 1$ depending only on $n$, such that any symplectomorphism $f : \mathbb{CP}^n \to \mathbb{CP}^n$ with*

$$\frac{1}{\Lambda} g \leq f^* g \leq \Lambda g$$

*is symplectically isotopic to a biholomorphic isometry of $\mathbb{CP}^n$ through the mean curvature flow.*

A theorem of Gromov [15] shows that, when $n = 2$, the statement holds true without any pinching condition by the method of pseudoholomorphic curves. Our theorem is not strong enough to give an analytic proof of Gromov's theorem for $n = 2$. However, for $n \geq 3$, this seems to be the first known result.

Smoczyk – Wang . 2002

**Theorem A.** *Let $\Sigma$ be a Lagrangian submanifold in $T^{2n}$. Suppose $\Sigma$ is the graph of $f : T^n \to T^n$ and the potential function $u$ of $f$ is convex. Then the mean curvature flow of $\Sigma$ exists for all time and converges smoothly to a flat Lagrangian submanifold.*

evolution eq for local potential $u$

$$\frac{du}{dt} = \frac{1}{\sqrt{-1}} \ln \frac{\det(I + \sqrt{-1} D^2 u)}{\sqrt{\det(I + (D^2 u)^2)}}.$$

from $\frac{d}{dt} u_i = g^{jk} u_{ijk}$

Chau - Chen - He , 2012

Consider the fully nonlinear parabolic equation for a function $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i=1}^n \arctan \lambda_i = \Theta \\ u(x, 0) = u_0(x) \end{cases}$$

here $\lambda_i$ : eigenvalues for Hess $u$

**Theorem 1.1** *Suppose that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a function with $L^\infty$ Hessian satisfying*

$$-(1-\delta)I_n \leq \text{ess inf } D^2 u_0 \leq \text{ess sup } D^2 u_0 \leq (1-\delta)I_n \tag{1.3}$$

a generalization of hypersurface case of Ecker - Huisken

*for any $\delta \in (0, 1)$. Then (1.1) has a longtime smooth solution $u(x, t)$ for all $t > 0$ with initial condition $u_0$ such that the following estimates hold:*

1. $-(1-\delta)I_n \leq D^2 u \leq (1-\delta)I_n$ *for all* $t > 0$.
2. $\sup_{x \in \mathbb{R}^n} |D^l u(x, t)|^2 \leq C_{l,\delta}/t^{l-2}$ *for all* $l \geq 3$, *and some* $C_{l,\delta}$ *depending only on* $l, \delta$.
3. *$u$ and $Du$ are Hölder continuous in time at $t = 0$ with Hölder exponents 1 and 1/2 respectively.*

*If in addition $|Du_0(x)| \to 0$ as $|x| \to \infty$, then $\sup_{x \in \mathbb{R}^n} |Du(x, t)| \to 0$ as $t \to \infty$. In particular, the graph $(x, Du(x, t))$ immediately becomes smooth and converges smoothly on compact sets to the coordinate plane $(x, 0)$ in $\mathbb{R}^{2n}$.*

$\Leftrightarrow$ $u$ stricly convex after changing coordinates

**Theorem 1.2** *Suppose that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a function with $L^\infty$ Hessian satisfying (1.3) and suppose that*

|H| control |A| & Use Bernstein Thm

$$\lim_{\lambda\to\infty} \lambda^{-2} u_0(\lambda x) = U_0(x) \tag{1.4}$$

*for some $U_0(x)$. Let $u(x,t)$ be the solution to (1.1) with initial data $u_0(x)$. Then $\lambda^{-2}u(\lambda x, \lambda^2 t)$ converges to a smooth self-expanding solution $U(x,t)$ to (1.1) uniformly on compact subsets of $\mathbb{R}^n \times (0,\infty)$ as $\lambda \to \infty$, and $U(x,t)$ converges to $U_0(x)$ uniformly on compact subsets of $\mathbb{R}^n$ as $t \to 0$. In particular, there is a one-to-one correspondence between self-expanding solutions to (1.1) satisfying (1.3) and Lipschitz functions on $\mathbb{R}^n$ which are homogeneous of degree 2 and satisfy (1.3).*

Chau - Chen - Yuan, 2013

**Theorem 1.1** *There exists a small positive dimensional constant $\eta = \eta(n)$ such that if $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ function satisfying*

$$-(1+\eta)I_n \le D^2 u_0 \le (1+\eta)I_n \tag{2}$$

almost complex potential

*then (1) has a unique longtime smooth solution $u(x,t)$ for all $t > 0$ with initial condition $u_0$ such that the following estimates hold:*

*(i)* $-\sqrt{3}I_n \le D^2 u \le \sqrt{3}I_n$ *for all $t > 0$.*
*(ii)* $\sup_{x\in\mathbb{R}^n} |D^l u(x,t)|^2 \le C_l/t^{l-2}$ *for all $l \ge 3$, $t > 0$ and some $C_l$ depending only on $l$.*
*(iii)* *$Du(x,t)$ is uniformly Hölder continuous in time at $t = 0$ with Hölder exponent $1/2$.*

# Monotonicity Formula (Huisken, 1990)

- Back heat kernel at $(x_0, t_0)$

$$\rho_{x_0,t_0}(x,t) = \frac{1}{\sqrt{4\pi(t_0-t)}^{\,n}} e^{\frac{-|x-x_0|^2}{4(t_0-t)}}, \quad t < t_0.$$

- If $\Sigma_t$ satisfies MCF, then

$(X - X_0)^\perp$

$$\frac{d}{dt}\int_{\Sigma_t} \rho_{x_0,t_0}\, d\mu_t = -\int_{\Sigma_t} \rho_{x_0,t_0} \left| H + \frac{1}{2(t_0-t)} X^\perp \right|^2 d\mu_t \le 0,$$

$\Rightarrow \lim_{t\to t_0} \int_{\Sigma_t} \rho_{x_0,t_0}\, d\mu_t$ exists, called the density at $(x_0, t_0)$.

- Have a similar formula for MCF in a Riemannian manifold and the density is also defined (White, 1997).

- $\int_{\Sigma_t} \rho_{x_0,t_0} d\mu_t$ is invariant under dilation. That is,

$$\int_{\Sigma_t} \rho_{x_0,t_0} d\mu_t = \int_{\Sigma_s^\lambda} \rho_{0,0} d\mu_s^\lambda.$$

- Choose $\lambda_i \to \infty$. If the limit of the flow of $\Sigma_s^{\lambda_i}$ exists, it must satisfy

$$H(x,s) - \frac{1}{2s} X_t^\perp = 0, \quad s < 0.$$

- For MCF in a Riemannian manifold, the blow up limit still lies in $\mathbb{R}^m$.
- $\Sigma$ is called a (normalized) self-shrinker in $\mathbb{R}^m$ if its position vector $X : \Sigma \to \mathbb{R}^m$ satisfies $H = -\frac{1}{2} X^\perp$. $\Rightarrow \sqrt{1-t}X$ satisfies MCF.

## Soliton Solution to MCF in $\mathbb{R}^m$

Solutions looks the same under MCF, either of the form $\varphi(t)X(x)$ or $X(x) + \psi(t)$ Direct computation shows that

- In case (1), $H = cX^\perp$ and $\varphi(t) = \sqrt{1 + 2ct}$ for a constant $c$. $c < 0$, solutions shrink, self-shrinkers, models for central blowup limits. $c = 0$, solutions do not move, minimal submanifolds. $c > 0$, solutions expand, called self-expanders, possible limit at infinity.
- In case (2), $H = T^\perp$ for a constant vector $T$ and $\psi(t) = tT$, called a translating solution. It is possible limit for max point blowup.
- Find such examples. Try to classify. Determine whether these examples are blowup limits of MCF. Use these examples to understand the singularities, and do surgery.

blow up

$\sigma > 0$

If $L(X,t)$ satisfies MCF, then $\tilde{\Sigma}(x,s) = \sigma\left(\Sigma(x, t_0 + \frac{s}{\sigma^2}) - y_0\right)$ also satisfies MCF.

若 $\Sigma$ defined in $[0,T)$

$\tilde{\Sigma}$ defined in $[-t_0\sigma^2, (T-t_0)\sigma^2)$

$$\left(\frac{\partial \tilde{\Sigma}}{\partial s}\right)^{\perp} = \left(\sigma \frac{\partial \Sigma}{\partial t} \cdot \frac{1}{\sigma^2}\right)^{\perp}$$

$$= \frac{1}{\sigma}\vec{H}(\Sigma(x,t)) = \vec{H}(\tilde{\Sigma}(x,s))$$

① Type I blow up ( Central pt blow up)

assume $\Sigma$ has an isolated singularity at $(y_0, T)$

$\sigma_i \to \infty$. $\tilde{\Sigma}^i(x,s) = \sigma_i\left(\Sigma(x, T + \frac{s}{\sigma_i^2}) - y_0\right)$

$-\sigma_i^2 T \le s < 0 \to$ anciant sol

$\exists$ a subseq.

$\ni \tilde{\Sigma}^i$ has Brakke flow limit $\tilde{\Sigma}^{\infty}$

Different subseq may have different limits

② Type II blow up (max pt blow up)

define $\tilde{\Sigma}^i(x, s) = \sigma_i \left( \Sigma\left(x, t_i + \frac{s}{\sigma_i^2}\right) - y_i \right)$, $\sigma_i \to \infty$

$y_i = \Sigma(x_i, t_i)$, $t_i \to T$. & $y_i \to y_0$

$-t_i \sigma_i^2 \leq s \leq (T - t_i) \sigma_i^2$

$\nexists\ (T - t_i)\sigma_i^2 \to \infty \rightsquigarrow$ eternal sol

$\exists$ a subseq $\ni \tilde{\Sigma}^i$ has a Brakke flow limit $\tilde{\Sigma}^\infty$

usually take $\sigma_i \sim \max \|A(x, t_i)\| = A(x_i, t_i)$

a singularity is called type I singularity if

$$\max_{\Sigma_t} |A|^2 \le \frac{C}{T-t}$$

- MCF in KE. Lag condition is preserved for cpt smooth sol. Smoczyk 1996

follow from $\frac{d}{dt}|\omega|^2 \le \Delta|\omega|^2 + c|\omega|^2$

Wang. Chen-Li. Neves.

No type I singularity for LMCF of graded Lag

- cpt. + almost Calibrated.
- Entire graph.
- also for MCF of symplectic surface in KE $\omega > 0$

Main reason: all smooth graded Lag shrinkers must be planes

From more general form of Huisken's monotonicity formula

<u>Thm</u> Let $f_t$ be a smooth family of functions on $\Sigma_t$

Then assuming all quantities are finite.

$$\frac{d}{dt}\int_{\Sigma_t} f_t\, \Phi(x_0,T)\, d\mathcal{H}^n = \int_{\Sigma_t}\left(\frac{df_t}{dt} - \Delta f_t\right)\Phi(x_0,T)\, d\mathcal{H}^n$$

$$- \int_{\Sigma_t} f_t \left| H + \frac{(X-x_0)^{\perp}}{2(T-t_0)}\right|^2 \Phi(x_0,T)\, d\mathcal{H}^n$$

$$\int \; z \;\; \Phi(x_0,T)(x,t) = \frac{\exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)}{(4\pi(T-t))^{n/2}}. \qquad \Theta_t(x_0,l) = \int_{M_t}\Phi(x_0,l)(x,0)\,d\mathcal{H}^k.$$

**Theorem 3.1** (White's Regularity Theorem). *There are $\varepsilon_0 = \varepsilon_0(n,k)$, $C = C(n,k)$ so that if $\partial M_t \cap B_{2R} = \emptyset$ and* $M_t$ satisfies MCF

$$\Theta_t(x,l) \leq 1+\varepsilon_0 \quad \text{for all} \quad l \leq R^2, x\in B_{2R}, \text{ and } t \leq R^2,$$

*then the $C^{2,\alpha}$-norm of $M_t$ in $B_R$ is bounded by $C/\sqrt{t}$ for all $t \leq R^2$.*

Being graded Lag. we have an extra function $\theta$ preseved under LMCF (assuming bd Lag)

$$\frac{d}{dt}\theta_t = \Delta \theta_t \quad \rightsquigarrow \quad \frac{d}{dt}\theta_t^2 = \Delta \theta_t^2 - 2|H|^2$$

L. smooth graded Lag shrinker. $H = -\frac{1}{2}X^\perp$ (assuming $\theta$ bded)

$L_t = \sqrt{-t}\,L$ satisfies LMCF &

$$H_t = \frac{1}{\sqrt{-t}} H = -\frac{1}{2}\frac{1}{\sqrt{-t}} X^\perp = -\frac{1}{2}\frac{1}{\sqrt{-t}}\frac{1}{\sqrt{-t}} X_t^\perp = -\frac{1}{2}\frac{X_t^\perp}{-t}.$$

Define $\Theta(t) = \int_{L_t} \underbrace{\theta^2 \Phi(0,0)\, d\mathcal{H}^n}$ $\longrightarrow$ scaling invariant

$$\Rightarrow 0 = \frac{d\Theta}{dt} = \int_{L_t} \left(\frac{d}{dt} - \Delta\right)\theta^2\, \Phi(0,0)\, d\mathcal{H}^n - \int_{L_t} \theta^2 \left|H - \frac{X_t^\perp}{2t}\right|^2 \Phi(0,0)\, d\mathcal{H}^n$$

$$= -2\int_{L_t} |H_t|^2\, \Phi(0,0)\, d\mathcal{H}^n \Rightarrow H_t = 0 \quad \text{minimal cone}$$

$\because$ L smooth $\therefore$ plane