Lecture 1

An introduction to Lagrangian Geometry

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Lagrangian submanifolds

1. $\mathbb{R}^{2n}$, $\omega_0 = \sum dx^j \wedge dy^j$, $\Omega = dz^1 \wedge \cdots \wedge dz^n$ $\quad z_j = x_j + i y_j$,

L^n n-dimal submanifold is called Lag. if $\omega|_L = 0$

Ex: (a) $x^1 \cdots x^n$ plane

(b) $f: \mathbb{R}^n \longmapsto R$,

the graph of $(x, \nabla f) = (x_1, \cdots\cdots x_n, \frac{\partial f}{\partial x^1}, \cdots\cdots, \frac{\partial f}{\partial x^n})$

$$\omega|_L = dx^j \wedge \frac{\partial^2 f}{\partial x^j \partial x^k} dx^k = 0$$

If L is Lag. $e_1, \cdots. e_n$. o.n basis at $T_x L$

$e_1, Je_1, \cdots . e_n, Je_n$ o.n basis for $\mathbb{R}^{2n}$

$$\Omega(T_x L) = \det_{\mathbb{C}}(e_1, \cdots e_n) = e^{i\theta}$$

$\theta : L \longrightarrow \frac{\mathbb{R}}{2\pi}$ is called Lag angle

$H = J\nabla\theta$

$$\Omega(e_1, e_2, \cdots, e_n) = e^{i\theta}$$

apply $e_j \Rightarrow i\, e_j(\theta)\, e^{i\theta} = \Omega(e_1, \cdots \bar{\nabla}_{e_j} e_k, \cdots, e_n)$

other terms give 0 $\quad = \Omega(e_1, \cdots, \langle \bar{\nabla}_{e_j} e_k, Je_k\rangle Je_k, \cdots, e_n)$

$$= \langle \bar{\nabla}_{e_j} e_k, Je_k\rangle\, i e^{i\theta}$$

Note $\langle \bar{\nabla}_{e_j} e_k, Je_k\rangle = \langle \bar{\nabla}_{e_k} e_j, Je_k\rangle = -\langle e_j, \bar{\nabla}_{e_k} Je_k\rangle$

$$= -\langle e_j, J\bar{\nabla}_{e_k} e_k\rangle = \langle Je_j, \bar{\nabla}_{e_k} e_k\rangle = \langle H, Je_j\rangle$$

$$\Rightarrow e_j(\theta) = \langle \nabla\theta, e_j\rangle = \langle J\nabla\theta, Je_j\rangle = \langle H, Je_j\rangle$$

$\therefore H = J\nabla\theta$ $\quad$ & if define $h_{ij}^k = \langle \bar{\nabla}_{e_i} e_j, Je_k\rangle$

fully symmetric

$H = 0 \iff \theta = \text{constant } \theta_0$ special Lag. of phase θ_0

Be called special Lag. because from the notion of Calibration

<u>Def</u>: a closed p-form α in (M, g) is called a calibration if its comass is 1. i.e $\alpha(v_1 \wedge \cdots \wedge v_p) \leq 1$ & "=" achived somewhere

$\Sigma^p \subset M$ is calibrated by α if $\alpha|_\Sigma = \text{Vol}_\Sigma$

i.e $\alpha(T_x \Sigma) = 1 \quad \forall x \in \Sigma$

<u>Prop</u> If Σ is a closed calibrated submfd, then Σ has the least volume in its homology class

Pf: Suppose $[\Sigma'] = [\Sigma]$. (exact case)

$$\text{Vol}(\Sigma) = \int_\Sigma \text{Vol}_\Sigma = \int_\Sigma \alpha = \int_{\Sigma'} \alpha \leq \int_{\Sigma'} \text{Vol}_{\Sigma'} = \text{Vol}\,\Sigma' \quad \#$$

(needs not to be smooth)

Re (Ω) is a calibration $\qquad$ Re ($e^{-i\theta_0}\Omega$)

$\because$ $|\Omega(v_1, \cdots, v_n)| \leq 1$ $\qquad$ (Ω is exact)

$\Rightarrow$ min Lag are vol minimizing

② all these can be defined in a Calabi-Yau mfd

(M, J, ω, Ω) $\quad$ ω: Kähler form. $\quad$ $\omega(u, v) = g(Ju, v)$

Ω: a parallel hol. $(n,0)$ form.

Lag. $\quad$ Lag angle. $\quad$ $H = J\nabla\theta$. , Re Ω a calibration

sLags play an importance role in string theory

We know little on the general existence of SLag

LMCF is a natural & potential way for constructing sLag

③ The notion of Lag is from symplectic manifold N^{2n}

(with a closed non-degenerate 2-form ω)

ex: cotangent bundle $N = T^*M$, $\hat{\omega} = \sum dp^i \wedge dq^i$

one form $\alpha = \sum p^i dq^i$ is a Lag submfd of N if $d\alpha = 0$

q^i coordinates for M

On symplectic mfd

The map: $TN \longmapsto T^*N$. is an isomorphism

$$v \longmapsto \alpha_v(\cdot) =: \omega(v, \cdot) = i_v \omega$$

If $\varphi_t : L \hookrightarrow N$ Lag, $\varphi_0 = id$, $\frac{d}{dt}\varphi_t\Big|_{t=0} = V$

then $d\alpha_v = 0$

$$\varphi_t^* \omega = 0 \Rightarrow 0 = \frac{d}{dt}\varphi_t^* \omega\Big|_{t=0} = \mathcal{L}_v \omega$$

$$= i_v d\omega + d\, i_v \omega$$

Recall. a Lag in a C.Y. have $H = J\nabla\theta$

$\rightsquigarrow \quad \alpha_H = \omega(H,\cdot) = g(JH,\cdot) = -\langle\nabla\theta,\cdot\rangle = -d\theta$

H is an infinitesimal Lag deformation V.F.

Expect MCF preserve Lag condition

(although $\theta : L \to \mathbb{R}/2\pi$, defined up to 2π; OK for $\nabla\theta$, $d\theta$)

If θ can be lifted to $\mathbb{R}$. valued, L is called graded Lag

$\boxed{\text{Darboux coordinates}}$: for $\forall p$ in Symplectic mfd N.

$\exists$ local coordinates (X,Y) near p. $\ni$ $\omega = \sum dx^i \wedge dy^i$

$\boxed{\text{Lag Nbh Thm}}$: L Lag in N. then $\exists$ an open Nbh U of L

in N. and an open Nbh U' of L in T^*L, and

an isomorphism $\varphi : U' \to U$, $\ni$ $\varphi^*(\omega) = \hat{\omega} = \sum dp^j \wedge dq^j$

(Exercise 1: Check these 2 Thms)

$\Rightarrow$ nearby Lag can be written as the graph of closed 1-form

(C^1-close).

In particular, for function f on L, the graph of df

(if in U') gives nearby Lag

$\Rightarrow$ Expect and hope Lag can have similar features

as hypersurface.

- Graphs of symplectomorphisms $\varphi : N \mapsto N$. $\varphi^*(\omega) = \omega$

  Give Lag submanifolds in $(N, \omega) \times (N, -\omega)$

③ Kähler mfds (N, J, ω) are symplectic mfds

- L Lag $\Rightarrow T^\perp L = JTL$

  $\omega(u, v) = g\langle Ju, v\rangle$ Can consider min Lag submfd

Here we also have the 2nd f.f on a Lag

$h(u, v, w) =: \langle \bar{\nabla}_v u, Jw \rangle$ is totally symmetric

- For a cpt min Lag immersion in Kähler

$$\left.\frac{d^2 A_t}{dt^2}\right|_{t=0} = \int_L \left(|d\alpha_V|^2 + |\delta \alpha_V|^2 - \overline{Ric}(V, V) \right) dVol_L$$

Exercise 2 : prove the above formula

$\Rightarrow$ ① If $\overline{Ric} \leq 0$, stable (strict if $\overline{Ric} < 0$)

② If $\overline{Ric} = 0$, the Jacobi field $\leftrightarrow$ α_V harmonic 1-form

McLean : these Jacobi fields are unobstructed. (☆)

i.e can be realized as the deformation of SLag.

SLag: smooth local moduli space of $\dim = b^1(L; \mathbb{Z})$

③ If $\overline{Ric} > 0$ & $b^1(L, \mathbb{Z}) \neq 0$. unstable, but can consider deformation only with $\alpha_V = df$. Hamiltonian deformation

Hamilton stable iff $\lambda_1(\Delta_g) \geq C$ (Oh)

From minimal submanifolds point of view. we are also interested in minimal Lagrangians

for Lag in Kähler. we have $d\alpha_H|_L = \mathcal{R}ic|_L$ ☆.

$\mathcal{R}ic$: Ric form defined by $\mathcal{R}ic(U, V) = Ric(JU, V)$

$\Rightarrow$ ⓐ On a min Lag $\omega|_L = 0$ & $\mathcal{R}ic|_L = 0$

hard to expect

ⓑ On Kähler-Einstein $\mathcal{R}ic = c\,\omega$.

These two conditions coincide in KE. So restrict the search for min Lag in KE.

- although we cannot defined θ, and have $\alpha_H = -d\theta$ on Lag in KE, $d\alpha_H|_L = Ric|_L = c\,\omega|_L = 0$

  $\longrightarrow$ MCF infinitesimal Lag deformation.

Study the existence of minimal Lag / SLag

- Direct construction
- GMT
- Deformation
- Gluing
- LMCF

(Thomas - Yau Conj, 2002)

Let L be a graded Lag in a CY, and satisfying some stable conditions. Then the LMCF of L will exist for all time and converges to the unique sLag in its Hamiltonian class

(Neves '2013)

Σ an embedded Lag in CY (surface). Then there exist L Hamiltonian isotopy to Σ, and the LMCF of L develops a finite time singularity.

(Joyce '2014)

TY conjecture is refined. Fukaya Categories & Bridgeland stability. Surgeries will be needed & unobstructed Lag