

## Uniqueness Theorems for Free Boundary Minimal Disks in Space Forms

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We show that a minimal disk satisfying the free boundary condition in a constant curvature ball of any dimension is totally geodesic. We weaken the condition to parallel mean curvature vector in which case we show that the disk lies in a three dimensional constant curvature submanifold and is totally umbilic. These results extend to higher dimensions earlier three dimensional work of J. C. C. Nitsche and R. Souam.

### 1 Introduction

In this short note, we consider the free boundary minimal disks in a ball in Euclidean space or a space  $N^n$  of constant curvature. These are proper branched minimal immersions of a disk into the ball that meet the boundary orthogonally (the conormal vector of the disk is normal to the boundary of the ball). Such surfaces have been extensively studied and they arise as extremals of the area functional for relative cycles in the ball. They also arise as extremals of a certain eigenvalue problem [3, 4]. We show here that any such free boundary minimal disk is a totally geodesic disk passing through the center of the ball.

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We extend the result to free boundary disks with parallel mean curvature in a ball. If the mean curvature is not zero, we show that such disks are contained in a totally geodesic three dimensional submanifold (an affine subspace in the flat case) and that the disk is totally umbilic.

Both of these results are known in case  $n=3$ . For the euclidean case, both the minimal and constant mean curvature results were obtained by Nitsche [5]. In the case of space forms, the results were obtained for  $n=3$  by Souam [7], and were generalized to constant contact angle boundary conditions by Ros and Souam [6].

The method of proof for the results is complex analytic as in the case  $n=3$ . The general idea goes back to H. Hopf and involves the construction of holomorphic differentials which then vanish under suitable conditions on the surface and suitable boundary conditions. In the  $n=3$  case, the Hopf differential is a holomorphic quadratic differential and the free boundary condition implies that it is real along the boundary in the sense that it has real values when applied to the unit tangent vector. Geometrically, this follows from the condition that the boundary be a principal curve. In higher dimensions the corresponding differential has values in the normal bundle, but there is an associated complex-valued quartic differential gotten by squaring the Hopf differential using the inner product. Our result is obtained by showing vanishing of this quartic differential and discussing the consequences.

There is an analog between minimal submanifolds of spheres and free boundary minimal submanifolds of the ball. Under this analog the Nitsche theorem is analogous to a result of Almgren [1] which says that a minimal  $\mathbb{S}^2$  in  $\mathbb{S}^3$  is totally geodesic. It is well known (see Calabi [2]) that there are many minimal immersions of  $\mathbb{S}^2$  in  $\mathbb{S}^n$  for  $n \geq 4$  which are not totally geodesic. From the point of view of this analogy our results are surprising. To make a more definite connection between the two problems, we point out that our results imply that any minimal  $\mathbb{S}^2$  in  $\mathbb{S}^n$  which is invariant under reflection through a totally geodesic  $\mathbb{S}^{n-1}$  must be totally geodesic.

## 2 Uniqueness Results

**Theorem 2.1.** Let  $u: D \rightarrow B^n$  be a proper branched minimal immersion such that  $u(D)$  meets  $\partial B$  orthogonally. Then,  $u(D)$  is an equatorial plane disk.  $\square$

**Proof.** Since  $u: D \rightarrow B^n$  is a minimal immersion,  $u$  is harmonic,  $u_{zz} = 0$ , and conformal,  $u_z \cdot u_z = 0$ . Since  $u$  is harmonic, it follows that

$$(u_{zz} \cdot u_{zz})_{\bar{z}} = 0.$$

Note that the normal bundle is smooth across branch points, and

$$u_{zz} = u_{zz}^\perp + \frac{u_{zz} \cdot u_z}{|u_z|^2} u_z + \frac{u_{zz} \cdot u_{\bar{z}}}{|u_{\bar{z}}|^2} u_{\bar{z}}, \tag{2.1}$$

where  $u_{zz}^\perp$  denotes the component of  $u_{zz}$  orthogonal to  $\Sigma = u(D)$ . But

$$u_{zz} \cdot u_z = \frac{1}{2}(u_z \cdot u_z)_z = 0$$

since  $u$  is conformal, and thus by (2.1)

$$u_{zz}^\perp \cdot u_{zz}^\perp = u_{zz} \cdot u_{zz}.$$

In polar coordinates  $(r, \theta)$  on the disk, we have

$$u_{zz}^\perp = \frac{1}{4} e^{-2i\theta} \left[ u_{rr}^\perp - \frac{1}{r^2} u_{\theta\theta}^\perp - \frac{2i}{r} u_{r\theta}^\perp \right]. \tag{2.2}$$

By the free boundary condition,  $u_r$  is orthogonal to  $\partial B$  along the boundary of the disk, and so  $u_r = \lambda u$  for some function  $\lambda$  on  $\partial D$ . Therefore on  $\partial D$ ,

$$u_{r\theta} = \lambda_\theta u + \lambda u_\theta = \frac{\lambda_\theta}{\lambda} u_r + \lambda u_\theta,$$

and so  $u_{r\theta}^\perp = 0$  on  $\partial D$ . It then follows from (2.2) that  $z^A(u_{zz}^\perp)^2$  is real on  $\partial D$ . Since  $z^A(u_{zz}^\perp)^2$  is holomorphic on  $D$ ,  $z^A(u_{zz}^\perp)^2$  must be constant. But  $z^A(u_{zz}^\perp)^2$  vanishes at the origin, and so  $z^A(u_{zz}^\perp)^2 \equiv 0$ .

In particular, on  $\partial D$  we have

$$0 = z^A(u_{zz}^\perp)^2 = \frac{1}{16}(u_{rr}^\perp - u_{\theta\theta}^\perp)^2.$$

By the minimality, this implies that  $u_{rr}^\perp = u_{\theta\theta}^\perp = 0$  on  $\partial D$ . Therefore, the second fundamental form of  $\Sigma$  is zero on  $\partial \Sigma$ . It follows that the second fundamental form of  $\partial \Sigma$  in  $S^{n-1}$  is zero. Therefore,  $\partial \Sigma$  is a great circle, and  $\Sigma$  must be an equatorial plane disk. ■

More generally, we have the following theorem.

**Theorem 2.2.** Let  $u: D \rightarrow B$  be a proper branched immersion with parallel mean curvature vector, from the disk  $D$  to a ball  $B$  in an  $n$ -dimensional space  $N$  of constant curvature, such that  $u(D)$  meets  $\partial B$  orthogonally. Then,  $\Sigma$  is contained in a three dimensional totally geodesic submanifold, and is totally umbilic. □

**Proof.** Let  $z = x_1 + ix_2$  be the standard complex coordinates on the disk  $D$ , and  $\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right)$ ,  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)$ . Let  $\nabla$  be the pull back connection in the pull back bundle  $u^*(TN)$ , let  $\nabla^\perp$  be the connection in the pull back  $u^*(N\Sigma)$  of the normal bundle to  $\Sigma = u(D)$  in  $N$ , and let  $A$  be the second fundamental form of  $\Sigma$  in  $N$ . Consider

$$A \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = \nabla_{\frac{\partial}{\partial z}}^\perp du \left( \frac{\partial}{\partial z} \right) = \frac{1}{4} [(A_{11} - A_{22}) - 2iA_{12}], \tag{2.3}$$

where  $A_{ij} = A \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$ ,  $i, j = 1, 2$ , are the coefficients of the second fundamental form in the coordinate basis. Then,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial z}}^\perp A \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) &= \frac{1}{8} [A_{11,1} - A_{22,1} + 2A_{12,2} + i(A_{11,2} - A_{22,2} - 2A_{12,1})] \\ &= \frac{1}{8} [-2A_{22,1} + 2A_{12,2} + i(2A_{11,2} - 2A_{12,1})] \\ &= 0, \end{aligned}$$

where the second equality follows since  $A_{11,i} + A_{22,i} = 0$ ,  $i = 1, 2$  by the assumption that  $\Sigma$  has parallel mean curvature, and the last equality follows from the Codazzi equations since the ambient manifold has constant sectional curvature. Therefore, the function

$$\phi(z) := A \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \cdot A \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right)$$

is holomorphic on  $D$ .

Consider polar coordinates  $(r, \theta)$  on the disk, so  $z = r e^{i\theta}$ . Set  $w = \log z = \log r + i\theta$ .

Then,

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z} \frac{\partial}{\partial w} = \frac{1}{z} \frac{\partial}{\partial w}$$

and

$$A \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = \frac{1}{z^2} A \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right).$$

On  $\partial D$  we have  $\frac{\partial}{\partial w} = \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta}$ , and

$$z^2 A \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = A \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) = A \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) - A \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right) - 2iA \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right). \tag{2.4}$$

By the free boundary condition, the outward unit normal  $\eta$  along  $\partial\Sigma$  is normal to the sphere. Thus, since  $du \left( \frac{\partial}{\partial r} \right) = \lambda \eta$ , where  $\lambda = |du \left( \frac{\partial}{\partial r} \right)|$ , we have

$$A \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right) = \nabla_{\frac{\partial}{\partial \theta}}^\perp du \left( \frac{\partial}{\partial r} \right) = \left[ -\lambda S \left( du \left( \frac{\partial}{\partial \theta} \right) \right) \right]^\perp,$$

where  $S$  is the shape operator of  $\partial B$ . But the shape operator of a sphere in a space of constant curvature is a multiple of the identity. Therefore,

$$A\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = 0 \quad \text{on } \partial D, \quad (2.5)$$

and, so  $z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)$  is real on  $\partial D$ .

Thus,  $z^A \phi(z)$  is a holomorphic function on  $D$  that is real on  $\partial D$ . This implies that  $z^A \phi(z)$  is constant. But  $z^A \phi(z)$  vanishes at the origin, and so  $z^A \phi(z) \equiv 0$ . In particular, by (2.4) and (2.5) we see that on  $\partial D$ ,

$$0 = z^A \phi(z) = \left[ A\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) - A\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) \right]^2.$$

Therefore,  $z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = 0$  on  $\partial D$ . But  $z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)$  is a holomorphic section of the normal bundle, and so  $z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \equiv 0$  on  $D$ . To see this, choose a global frame  $s_1, \dots, s_{n-2}$  for the normal bundle  $u^*(N\Sigma)$  over the disk  $D$ . Then,

$$z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \sum_{j=1}^{n-2} f_j s_j$$

for some complex-valued functions  $f_j$  on  $D$ ,  $j = 1, \dots, n-2$ , and

$$0 = \nabla_{\frac{\partial}{\partial \bar{z}}}^\perp \left( z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \right) = \nabla_{\frac{\partial}{\partial \bar{z}}}^\perp \left( \sum_{j=1}^{n-2} f_j s_j \right) = \sum_{j=1}^{n-2} \left( \frac{\partial f_j}{\partial \bar{z}} + \sum_{k=1}^{n-2} a_{kj} f_k \right) s_j,$$

where  $a_{jk}$ ,  $j, k = 1, \dots, n-2$ , are given by  $\nabla_{\frac{\partial}{\partial \bar{z}}}^\perp s_j = \sum_{k=1}^{n-2} a_{jk} s_k$ . Therefore, we have

$$\begin{aligned} \frac{\partial f_j}{\partial \bar{z}} + \sum_{k=1}^{n-2} a_{kj} f_k &= 0 \quad \text{on } D \\ f_j &= 0 \quad \text{on } \partial D \end{aligned} \quad \text{for } j = 1, \dots, n-2.$$

This implies that  $f_j \equiv 0$  on  $D$  for  $j = 1, \dots, n-2$ , and so  $z^2 A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \equiv 0$  on  $D$ .

Hence,  $A\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \equiv 0$  on  $D$ . By (2.3), this implies that  $A_{11} = A_{22}$  and  $A_{12} = 0$  on  $D$ , or  $A_{ij} = \frac{1}{2} H \delta_{ij}$ . Consider the three dimensional linear space  $\mathcal{V}(x)$  at each point  $x$  of  $D$  spanned by the tangent space  $T_{u(x)}\Sigma$  and the mean curvature vector  $H_{u(x)}$ . We claim that  $\mathcal{V}$  is parallel. To see this, let  $X \in \Gamma(u^*(TN))$  be a vector field along  $\Sigma$  such that  $X_x \in \mathcal{V}(x)$  for all  $x \in D$ . Then  $X = X^T + fH$ , where  $X^T$  is tangent to  $\Sigma$  and  $f$  is a smooth function

on  $D$ . Then, if  $Y \in \Gamma(u^*(T\Sigma))$ ,

$$\begin{aligned} \nabla_Y X &= (\nabla_Y X^T)^T + \nabla_Y^\perp X^T + X(f)H + f(\nabla_Y H)^T + f\nabla^\perp H \\ &= (\nabla_Y X)^T + \sum_{i,j=1}^2 c_{ij} A_{ij} + X(f)H + f(\nabla_Y H)^T \quad (\text{since } H \text{ is parallel}) \\ &= (\nabla_Y X)^T + cH + X(f)H + f(\nabla_Y H)^T \quad \left( \text{since } A_{ij} = \frac{1}{2} H \delta_{ij} \right) \\ &\in \mathcal{V}. \end{aligned}$$

Therefore,  $\mathcal{V}$  is parallel. It follows that  $\Sigma$  is contained in a three dimensional submanifold  $N_0$  of  $N$ . To see this, let  $p = u(0)$ , and let  $N_0$  be the totally geodesic three dimensional submanifold containing  $p$  and tangent to  $\mathcal{V}(p)$ . If we fix a basis  $e_1, e_2, e_3$  for  $\mathcal{V}(p)$ , then the submanifold  $N_0$  can be characterized as the set of points  $q$  of  $N$  which can be joined by a curve  $\gamma(t)$  from  $p$  to  $q$  such that  $\gamma'(t)$  is a linear combination of the parallel transports  $e_1(t), e_2(t), e_3(t)$  for each  $t$ . Since  $\mathcal{V}$  is parallel along  $\Sigma$  and contains the tangent space to  $\Sigma$  at each point, this property holds for all  $q \in \Sigma$ . Therefore,  $\Sigma \subset N_0$ .

The classification of totally umbilic surfaces in three dimensional space forms is well known (see [8]). In the case of  $\mathbb{R}^n$ ,  $\Sigma$  must be an equatorial plane or a spherical cap meeting  $\partial B$  orthogonally; in particular, if  $\Sigma$  is minimal,  $\Sigma$  must be an equatorial plane disk in the ball. In the case of  $\mathbb{S}^n$ ,  $\Sigma$  must be a spherical cap meeting  $\partial B$  orthogonally; in particular, if  $\Sigma$  is minimal, it must be a totally geodesic disk in  $B$ . In the case of hyperbolic space,  $\Sigma$  must be a totally geodesic plane disk or its equidistant, a horosphere, or a round sphere meeting  $\partial B$  orthogonally; in particular, if  $\Sigma$  is minimal, then  $\Sigma$  must be a totally geodesic plane disk in  $B$ . ■

**Corollary 2.3.** Let  $\Sigma$  be an immersed minimal  $\mathbb{S}^2$  in  $\mathbb{S}^n$  that is invariant under reflection through a totally geodesic  $\mathbb{S}^{n-1}$ . Then,  $\Sigma$  is totally geodesic. □

**Proof.** Since the surface  $\Sigma$  intersects the equatorial  $\mathbb{S}^{n-1}$  orthogonally, it is transversal. Thus, the intersection consists of smooth circles. If we consider an inner-most circle (one which bounds a disk disjoint from the other circles), then we get a free boundary minimal disk in a hemisphere, and by Theorem 2.2 this disk is totally geodesic. It follows that  $\Sigma$  is the corresponding totally geodesic minimal  $\mathbb{S}^2$  since it is assumed to be reflection symmetric. ■

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## References

- [1] Almgren, F. J. "Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem." *Annals of Mathematics. Second Series* 84, no. 2 (1966): 277–92.
- [2] Calabi, E. "Minimal immersions of surfaces in Euclidean spheres." *Journal of Differential Geometry* 1, no. 1–2 (1967): 111–25.
- [3] Fraser, A. and R. Schoen. "The first Steklov eigenvalue, conformal geometry, and minimal surfaces." *Advances in Mathematics* 226, no. 5 (2011): 4011–30.
- [4] Fraser, A. and R. Schoen. "Sharp eigenvalue bounds and minimal surfaces in the ball." (2012): preprint arXiv:1209.3789 [math.DG].
- [5] Nitsche, J. C. C. "Stationary partitioning of convex bodies." *Archive for Rational Mechanics and Analysis* 89, no. 1 (1985): 1–19.
- [6] Ros, A. and R. Souam. "On stability of capillary surfaces in a ball." *Pacific Journal of Mathematics* 178, no. 2 (1997): 345–61.
- [7] Souam, R. "On stability of stationary hypersurfaces for the partitioning problem for balls in space forms." *Mathematische Zeitschrift* 224, no. 2 (1997): 195–208.
- [8] Spivak, M. *A Comprehensive Introduction to Differential Geometry*, vol. 3 and 4. Publish or Perish: Boston, 1970.