Free boundary minimal surfaces and the Steklov eigenvalue problem

1. Let Σ be an immersed submanifold of dimension k in $\mathbb{R}^n$.

   (a) Show that Σ has zero mean curvature $H = 0$ if and only if the coordinate functions of Σ in $\mathbb{R}^n$ are harmonic, $\Delta_\Sigma x^i = 0$, $i = 1, \ldots, n$.

   (b) Assume Σ is minimal ($H = 0$). Show that $\Delta |x|^2 = 2k$

2. Let Σ^k be a free boundary minimal submanifold in B^n.

   (a) Show that $|\partial\Sigma| = k|\Sigma|$, where $|M|$ denotes the volume of M. (*Hint:* Use 1(b) and integrate.)

   (b) Show that $\partial\Sigma$ cannot lie strictly in a hemisphere. (*Hint:* Use 1(a) to show that the coordinate functions are 'balanced' on $\partial\Sigma$)

3. Consider the catenoid in $\mathbb{R}^3$ parametrized by

$$u(t, \theta) = (\cosh t \cos\theta, \cosh t \sin\theta, t)$$

   with $0 < \theta < 2\pi$, $t \in \mathbb{R}$.

   (a) Find the value T such that u restricted to $S^1 \times [-T, T]$ defines a free boundary minimal annulus in a ball, and determine the radius R of that ball.

   (b) Write down a parametrization of a free boundary minimal annulus in the unit ball B^3 by using a rescaling of the given parametrization using your result from part (a). This surface is referred to as the "critical catenoid".

   (c) Calculate the length of the boundary of the critical catenoid (from part (b)).

4. (Dirichlet-to-Neumann operator) Let $(M, \partial M)$ be a Riemannian manifold. Given a function $u \in C^\infty(M)$, let $\hat{u}$ be the harmonic extension of u:

$$\begin{cases} \Delta_g \hat{u} = 0 & \text{on } M, \\ \hat{u} = u & \text{on } \partial M. \end{cases}$$

   The Dirichlet-to-Neumann map is the map

$$L : C^\infty(\partial M) \to C^\infty(\partial M)$$

   given by

$$Lu = \frac{\partial \hat{u}}{\partial \eta}$$

   where η be the outward unit normal along ∂M.

   (a) Use Green's formulas to show that the Dirichlet-to-Neumann map is self-adjoint and non-negative.

Recall that the trace embedding from $W^{1,2}(M)$ to $L^p(\partial M)$ is bounded for $p \le \frac{2(n-1)}{n-2}$ for $n \ge 3$ and compact for $p < \frac{2(n-1)}{n-2}$ for all n. This makes it possible to diagonalize the quadratic form on the unit sphere in $L^2(\partial M)$ to construct the eigenvalues $\sigma_0,\ \sigma_1, \sigma_2, \ldots$ and orthonormal eigenfunctions $u_0,\ u_1,\ u_2, \ldots$ satisfying

$$\begin{cases} \Delta_g u_i = 0 & \text{on } M, \\ \frac{\partial u_i}{\partial \eta} = \sigma_i u_i & \text{on } \partial M. \end{cases}$$

Note that $\sigma_0 = 0$ and u_0 is a constant function while $\sigma_1 > 0$ (we take M to be connected).

Thus the eigenfunctions are critical points of the Rayleigh quotient

$$\frac{\int_M |\nabla u|^2 \, dv_M}{\int_{\partial M} u^2 \, dv_{\partial M}}.$$

among $W^{1,2}$ functions on M. The kth eigenvalue σ_k of L can be characterized variationally as:

$$\sigma_k = \inf\{\frac{\int_M |\nabla u|^2 \, dv_M}{\int_{\partial M} u^2 \, dv_{\partial M}} \ : \ \int_{\partial M} u u_i = 0,\ i = 0, 1, \ldots, k-1\}$$

where the functions are taken in $W^{1,2}(M)$. The eigenfunction is then a function which achieves the infimum. By elliptic regularity the eigenfunctions are smooth on the closure of M.

(b) When $k = 1$ carry this procedure out to construct σ_1 and u_1.

5. (a) Let (M, g) be a Riemannian manifold and $c > 0$. Show that $\sigma_k(cg) = \frac{1}{\sqrt{c}}\sigma_k(g)$.

   (b) Show that $\sigma_1(g)|\partial M|^{\frac{1}{n-1}}$ is scale invariant.

6. (Critical Möbius band)

   (a) Consider the Möbius band M as $\mathbb{R} \times S^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$. Show that

   $$\varphi(t, \theta) = (2 \sinh t \cos\theta, 2 \sinh t \sin\theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta)$$

   is a minimal embedding of M into $\mathbb{R}^4$.

   (b) Show that for a unique choice of T_0 the restriction of φ to $[-T_0, T_0] \times S^1$ is an embedding into a ball meeting the boundary orthogonally.

   (c) We may rescale the radius of the ball to 1 to get the *critical Möbius band*, an embedded free boundary minimal Möbius band in B^4. Determine the length of the boundary of the critical Möbius band.

7. (Hopf differential) Suppose Σ is a surface of constant mean curvature in $\mathbb{R}^3$, and let

   $$\varphi(z) := (h_{11} - h_{22}) - 2i h_{12}$$

   where $z = x + iy$ is a local complex coordinate on Σ, and h_{ij} are coefficients of the second fundamental form of Σ. Show that f is a holomorphic function.

8. (a) Let Σ be a free boundary minimal surface in B^3. Show that $\partial\Sigma$ is a principal curve in Σ

   (b) Let Σ be a free boundary minimal hypersurface in B^n. Show that $\partial\Sigma$ is umbilic in Σ (that is, the second fundamental form is proportional to the metric).

9. Let Σ be an equatorial plane disk in B^3. Show that Σ has index 1:

   (a) Show that Σ has index at least 1.

   (b) Use the fact that any equatorial plane disk in B^3 minimizes area subject to the constraint that it divides the volume of the ball in half, to show that Σ has index at most one.

10. Let Σ be an equatorial plane disk in B^n. Show that Σ has index $n-2$. (*Hint:* Observe that the Jacobi operator decouples in to scalar operators on the component functions.)