Topics in Scalar Curvature
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CHAPTER 1

Introduction and motivation

Let (M, g) be a smooth Riemannian manifold with Riemannian curvature tensor R_{ijkl}. In the study of differential geometry, there are three types of curvatures that has received a lot of attention.

The sectional curvature measures the curvature of two dimensional submanifolds of (M, g). In local coordinate, the sectional curvature of a two dimensional tangent plan spanned by e_i, e_j is given by R_{ijji}. The Ricci curvature is the trace of the sectional curvature. In local coordinate, the Ricci tensor is defined as $\mathrm{Ric}_{ij} = \sum_k R_{ikkj}$. The scalar curvature is the trace of the Ricci curvature: $R = \sum_{i,j} R_{ijji}$. Note that in our convention the scalar curvature of a two dimensional surface is twice its Gauss curvature.

The study of curvature dates back to the time of Gauss and Riemann, where curvature was first observed as the quantity that distinguish locally a general Riemannian manifold and the Euclidean space. Since then lots of studies have been carried out to understand the connection between the curvature with local and global geometric and topological aspects of the manifold. Here we just name a few of well known results of this flavor: the Gauss-Bonnet theorem, Toponogov comparasion theorem, the Bochner-Weizenböck formula, Bonnet-Myers theorem, Cheeger-Gromoll's volume comparasion theorem, Gromov's finiteness theorem, etc. From our very crude list one readily sees that results concerning scalar curvature are much fewer compared to those of the sectional or Ricci curvature. Indeed, scalar curvature measures the average the sectional curvature among all distinct two planes, and is believed to carry relatively few information about the manifold. This adds considerable delicacy into the study of the scalar curvature.

In this chapter we introduce three motivations to study the scalar curvature, and various phenamina unique to it.

1. Einstein equations of general relativity

The first senario where scalar curvature naturally occurs is the theory of general relativity. Suppose (S^{n+1}, g) is a Minkowski manifold representing the space time. Then it satisfies the Einstein equation, namely

$$\mathrm{Ric}_S - \frac{1}{2} R_S g = T,$$

where T is called the stress-energy tensor representing matter in the space time.

Assume that M^n is a submanifold of S^{n+1}. At a point in M, take a normal frame of S, $\{e_i\}_{i=0,\dots,n}$ such that e_0 is the unit normal vector of M. Restrict the Einstein equation on M with respect to e_0, e_0, we get

$$\mathrm{Ric}_{00} - \frac{1}{2} R_S = T_{00}.$$

Expanding in local coordinates, $\mathrm{Ric}_{00} = \sum_j R_{0jj0}$, and

$$\frac{1}{2} R_S = \sum_{0 \le i \le j \le n} R_{ijji} = \sum_j R_{0jj0} + \sum_{1 \le i \le j \le n} R_{ijji}.$$

Denote R^M the Riemannian tensor on M and h the second fundamental form of the embedding $M \subset S$. By the Gauss equation,

$$R_{ijji} = R^M_{ijji} + h_{ii}h_{jj} - h_{ij}^2.$$

Therefore

$$\frac{1}{2}R_S = \mathrm{Ric}_{00} + \frac{1}{2}R_M + \frac{1}{2}(\operatorname{tr} h)^2 - \frac{1}{2}\|h\|^2.$$

By the Einstein equation, one sees that

$$\frac{1}{2}(R_M) + (\operatorname{tr} h)^2 - \|h\|^2 = T_{00} := \mu. \tag{1.1}$$

In the theory of general relativity, $T_{00} = \mu$ is called the mass density. One may also derive other relations from the Einstein equation by taking the $0j$ component. By a similar calculation, we have

$$\operatorname{div}(h - (\operatorname{tr} h)g) = T_{0j}. \tag{1.2}$$

Together, equation 1.1 and 1.2 are called the **constraint equations**. They provide a necessary condition for a Riemannian manifold to be realized as a submanifold of a space time.

A particularly important case is when h is identically zero on M. In other words, M is a totally geodecis submanifold of S. The physical reason is that, when $h \equiv 0$, the reflection of the time variable $t \to -t$ is an isometry. It is called time symmetric space time. We see from the previous equation that in this case $R_M = 2\mu$. Note that since μ is the mass density, we always assume that μ is nonnegative. Then R_M is nonnegative. We then conclude that

Any Riemannian manifold satisfying the constraint equations must have nonnegative scalar curvature.

From this prospective, the study of Riemannian manifold with nonnegative scalar curvature can be embedded into the study of the constraint equations of general relativity.

2. Variation of total scalar curvature

On a Riemannian manifold (M, g), consider Einstein-Hilbert functional

$$\mathscr{R}(g) = \int_M R_g d\mu_g$$

on all Riemannian metrics with unit volume. It is shown by Hilbert that for $n \geq 3$ the critical points of this functional are Einstein metrics, i.e. metrics satisfying $\mathrm{Ric}(g) = c \cdot R(g)g$. This can be seen from the following calculation.

Let g be a Riemannian metric with unit volume which is a critical point of $\mathscr{R}$. Set $g_t = g + th$, $t \in (-\epsilon, \epsilon)$ be a smooth family of Riemannian metrics, h a smooth compactly supported $(0,2)$ tensor. Let $\bar{g}_t = \mathrm{Vol}(t)^{-2/n} g_t$ be normalized with unit volume. Then $\bar{g}_t$ is a valid variation. To calculate the derivative of $\mathscr{R}(\bar{g}_t)$, we first observe that in local coordinates,

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}),$$

$$\mathrm{Ric}_{ij} = \sum_k \left(\Gamma^k_{ij,k} - \Gamma^k_{ki,j} + \sum_l (\Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}) \right),$$

where the comma denotes a partial derivative in the given direction. We then see the derivatives of the Christoffel symbol and the Ricci curvature under this variation are

$$\dot{\Gamma}^k_{ij} = \frac{1}{2}\sum g^{kl}(h_{il,j} + h_{jl,i} - h_{ij,l}), \qquad \dot{\mathrm{Ric}}_{ij} = \sum_k (\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{ki,j}). \tag{2.1}$$

Therefore we find that $\dot{R}$ can be written as

$$\dot{R} = -\sum_{i,j} h^{ij}\operatorname{Ric}_{ij} + \text{divergence terms},$$

here $h^{ij} = \sum_{k,l} g^{ik}g^{jl}h_{kl}$. As for the volume form, one easily checks that

$$\frac{d}{dt}\operatorname{Vol}_{g_t} = \frac{1}{2}\operatorname{tr}_{g_t}(h)d\operatorname{Vol}_{g_t}.$$

Putting these together and use the divergence theorem, we find

$$\frac{d}{dt}\mathscr{R}(g_t) = -\int_M \left\langle h, \operatorname{Ric}(g_t) - \frac{1}{2}R_t g_t\right\rangle d\operatorname{Vol}_{g_t}.$$

Since $\mathscr{R}(\bar{g}_t) = \operatorname{Vol}(g_t)^{(2-n)/n}\mathscr{R}(g_t)$ we find that

$$\frac{d}{dt}|_{t=0}\mathscr{R}(\bar{g}_t) = -\operatorname{Vol}(g)^{(2-n)/n}\int_M \left\langle h, \operatorname{Ric}(g) - \frac{1}{2}R(g)g + \frac{n-2}{2n}\mathscr{R}(g)g\right\rangle d\operatorname{Vol}_{g_t}.$$

Therefore g is a critical point for $\mathscr{R}$ functional if and only if $\operatorname{Ric}(g) = cg$ for some constant c.

The existence and geometric properties of Einstein metrics have been long standing questions that have stimulated the development of many area. In dimension 2 and 3 Einstein metrics are metrics with constant sectional curvature, and can be handled by Ricci flow. In dimension 4 or above, relatively few is known. One major difficulty, among others, is that we do not know whether the extremal of the function $\mathscr{R}$ is local maximum or minimum.

3. Conformal geometry

Let's continue looking at the Einstein-Hilbert functional. As discussed previously, we do not know whether $\mathscr{R}$ has an infimum or supreme in general. However, we are going to see that $\mathscr{R}$ does attain a minimum when we restrict our consideration within a special family of metrics, namely, the **conformal** class of metrics.

Let g_0 be a background metric on a compact manifold M^n, $n \geq 3$. Define the conformal class of g_0 by

$$[g_0] = \{g = e^{2u}g_0 : u \in C^\infty(M)\}.$$

And the Yamabe invariant of this conformal class

$$\mathscr{Y}([g_0]) = \inf\{\mathscr{R}(g) : g \in [g_0], \operatorname{Vol}(g) = 1\}.$$

We are going to see shortly that $\mathscr{Y}([g_0])$ is finite for any smooth background metric g_0. For the convenience of calculation, let's use $u^{\frac{4}{n-2}}$ as the conformal factor, namely let $g = u^{\frac{4}{n-2}}g_0$. We have the following important formula that will be used throughout our discussion.

LEMMA 3.1 (Conformal formula). *The scalar curvature of $g = u^{\frac{4}{n-2}}g_0$ and the scalar curvature of g_0 are related by*

$$R(g) = -c(n)^{-1}u^{-\frac{n+2}{n-2}}Lu, \tag{3.1}$$

the constant $c(n) = \frac{n-2}{4(n-1)}$, L is an elliptic operator $Lu = \Delta_{g_0}u - c(n)R(g_0)u$, called the conformal Laplacian.

PROOF. Denote $f = \frac{2}{n-2}\log u$, then $g = e^{2f}g_0$. Let $\Gamma, R_{ijkl}, \operatorname{Ric}$, etc. denote the geometric quantities of the metric g and let $\Gamma_0, (R_0)_{ijkl}, \operatorname{Ric}_0$, etc. denote those of g_0. Take a local coordinate normal at one point. Then we first see the new Christoffel symbols are given by

$$\Gamma_{ij}^k = (\Gamma_0)_{ij}^k + \delta_i^k\partial_j f + \delta_j^k\partial_i f - g_{ij}\partial_k f.$$

By the previous formula relating the Christoffel symbol and the Ricci tensor,

$$\mathrm{Ric}_{ij} = (\mathrm{Ric}_0)_{ij} - (n-2)[\partial_i\partial_j f - (\partial_i f)(\partial_j f)] + (\Delta_0 f - (n-2)\|\nabla f\|^2)(g_0)_{ij}.$$

Taking trace we get

$$R = e^{-2f}(R_0 + 2(n-1)\Delta f - (n-2)(n-1)\|\nabla f\|^2).$$

Now replace $f = \frac{2}{n-2}\log u$, we obtain the desired formula for the scalar curvature. □

Thus for a conformal metric $g = u^{\frac{4}{n-2}} g_0$ with unit volume, its total scalar curvature is given by

$$\mathscr{R}(g) = c(n)^{-1}\int_M |\nabla_{g_0} u|^2 + c(n)R(g_0)u^2 d\,\mathrm{Vol}_{g_0}.$$

Therefore the Yamabe invariant is a well defined real number. Regarding the sign of the Yamabe invariant, we have the follwoing trichotomy theorem.

THEOREM 3.2. *Let (M^n, g_0) be a closed compact smooth Riemannian manifold with $n \geq 3$. Then the conformal class of g_0 belongs to one of the following three classes:*

(1) $\mathscr{Y}([g_0]) > 0 \Leftrightarrow \exists g \in [g_0], R(g) > 0 \Leftrightarrow \lambda_1(-L) > 0.$
(2) $\mathscr{Y}([g_0]) = 0 \Leftrightarrow \exists g \in [g_0], R(g) = 0 \Leftrightarrow \lambda_1(-L) = 0.$
(3) $\mathscr{Y}([g_0]) < 0 \Leftrightarrow \exists g \in [g_0], R(g) < 0 \Leftrightarrow \lambda_1(-L) < 0.$

PROOF. The theorem is an easy consequence of the variational characterization of $\lambda_1(-L)$:

$$\lambda_1(-L) = \inf_{u\in C^\infty(M)} \frac{c(n)^{-1}\int_M |\nabla u|^2 + c(n)R(g_0)u^2}{\int_M u^2}$$

and the fact that the first eigenfunction is always positive. □

As a corollary, we have

COROLLARY 3.3. *Let (M^n, g_0) be a compact smooth Riemannian manifold with $n \geq 3$. If $\mathscr{R}(g_0) < 0$ then $\mathscr{Y}([g_0]) < 0$.*

PROOF. Take $u \equiv 1$ into the variational characterization of $\lambda_1(-L)$, we find that $\lambda_1(-L) < 0$. By the above theorem, $\mathscr{Y}([g_0]) < 0$. □

4. Manifolds with negative scalar curvature

A natural question in the study of scalar curvature is what topological consequences we may obtain from scalar curvature conditions. As we will see in this section, negative scalar curvature does not have any topological obstruction. Namely, we are going to prove

THEOREM 4.1. *Any smooth compact manifold M^n, $n \geq 3$ has a metric with negative scalar curvature.*

The following basic example turns out to be important in constucting new manifolds with controlled scalar curvature.

EXAMPLE 4.2 (Schwarzschild metric). *On a manifold $x \in \mathbb{R}^n - \{0\}$, $n \geq 3$, define $g_{ij} = (1 + \frac{m}{2|x|^{n-2}})^{4/n-2}\delta_{ij}$, m is a positive real number. Since $\frac{1}{|x|^{n-2}}$ is the Green's function of Laplacian, we see that the scalar curvature of g is everywhere zero. The hypersurface determined by $|x| = m/2$ is a totally geodesic submanifold, called the horizon. Now reflect the part $|x| > m/2$ across this horizon, we obtain a complete smooth Riemannian manifold with zero scalar curvature. It is diffeomorphic to an annulus $S^{n-1} \times (0,1)$.*

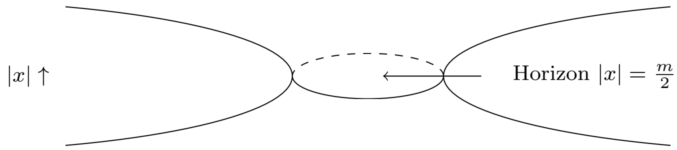

FIGURE 1. Doubled Schwarzschild

Near $|x| \approx \infty$ the Schwarzschild metric converges smoothly to the Euclidean metric. We can use this property to constuct a metric on the connected sums of any two Riemannian manifolds while keeping the total scalar curvature arbitrarily close to their sums. To see this, let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $p_1 \in M_1$, $p_2 \in M_2$ be two points on them. For any $\epsilon > 0$, pick a very small radius δ, so that the geodesic ball of radius δ around p_1 and p_2 contains arbitrarily small total scalar curvature: $\int_{B_\delta(p_1)} |R(g_1)| < \epsilon$, $\int_{B_\delta(p_2)} |R(g_2)| < \epsilon$. Take a very large number R and scale Schwarzschild inside $B_R(0)$ down such that its two ends have radius $\delta/2$. Then use a cutoff function to glue this very small Schwarzschild neck to $M_1 - B_{\delta/2}(p_1)$ and $M_2 - B_{\delta/2}(p_2)$, leaving elsewhere the same metric. Note that in this procedure, the total scalar curvature changes by at most ϵ, provided that the the metric outside $B_R(0)$ on the Schwarzschild is ϵ-close to the Euclidean metric in the C^2 sense.

We therefore conclude the following

PROPOSITION 4.3. *For any two smooth compact Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2), *and any* $\epsilon > 0$, *there exists a metric* g *on* $M_1 \# M_2$ *such that*

$$|\mathscr{R}(g)| - \mathscr{R}(g_1) - \mathscr{R}(g_2)| < \epsilon.$$

As a special case, we note that taking the connect sum with a sphere of the same dimension does not change the topology of the manifold. This observation leads to the following

COROLLARY 4.4. *Let* $n \geq 3$. *If* S^n *has a metric with negative total scalar curvature then so does every other* n *dimensional manifold.*

PROOF. Fix a metric g_0 on S^n with negative total scalar curvature. For any Riemannian manifold (M^n, g) choose $\lambda > 0$ large such that $\mathscr{R}(\lambda^2 g_0) = \lambda^{n-2} < |\mathscr{R}(g)|$. Then take the connected sum of $(S^n, \lambda^2 g_0)$ and (M, g). $\square$

In the remaining of this section we are going to conctruct a metric on S^n with negative total scalar curvature. To do so we first discuss a general smoothening technique that produces large family of metric with prescribed total scalar curvature.

The simple case that motivates the construction is as follows.

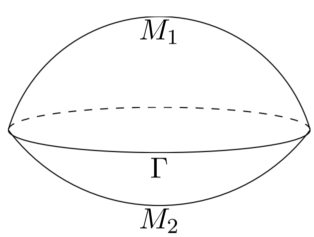

FIGURE 2. Gluing two surfaces sharing a common boundary

Let M_1, M_2 be two identical pieces of a spherical cap of constant Gauss curvature reflective along a common boundary curve Γ, and $M = M_1 \cup M_2$. Then by the Gauss-Bonnet theorem,

$$\int_{M_1} K + \int_{M_2} K + 2\int_\Gamma k = \int_M K = 4\pi.$$

Therefore the convex curve Γ contributes positively to the total curvature of M. Conversely if Γ is a concave curve then $\int_\Gamma k$ contributes negatively to the total curvature.

In general, consider two manifolds M_1^n, M_2^n joining along a common hypersurface Σ, $M = M_1 \cup M_2$, $\partial M_1 = \partial M_2 = \Sigma$. Σ has two mean curvatures in M_1 and M_2, denoted by H_1, H_2, respectively. Here our convention is that the standard unit sphere in the Eucliden space has constant mean curvature 1. Then we actually have that

$$\int_M R(g) \approx \int_{M_1} R(g_1) + \int_{M_2} R(g_2) + 2\int_\Sigma (H_1 + H_2)$$

in the sense that there is a smooth metric g on M such that its total scalar curvature is arbitrarily closed to the right hand side. In fact, if $H_1 + H_2 > 0$ pointwise then one can actually smoothen the metric on M to make its scalar pointwise positive. See [**Mia02**]. The motivation behind it is following. Suppose M is a smooth Riemannian manifold. Let $\nu = e_n$ be the unit normal vector of Σ pointing into M_1, t be the coordinate such that $t =$ constant representing hypersurfaces of constant distance to Σ. Then by the Gauss equation,

$$\begin{aligned} R_M - 2\operatorname{Ric}(\nu,\nu) &= \sum_{i,j=1}^{n-1} R^M_{ijji} \\ &= \sum_{i,j=1}^{n-1} (R^\Sigma_{ijji} - h_{ii}h_{jj} + h^2_{ij}) \\ &= R_\Sigma - H^2 + |h|^2 \end{aligned}$$

$$\Rightarrow R_M = R_\Sigma - H^2 + |h|^2 + 2\operatorname{Ric}(\nu,\nu).$$

On the other hand, the derivative of the mean curvature assuming Σ moves with unit speed ν is given by

$$\frac{\partial H}{\partial t} = -\operatorname{Ric}(\nu,\nu) - |h|^2.$$

Therefore we conclude that

$$R_M = R_\Sigma - (|h|^2 + H^2) - 2\frac{\partial H}{\partial t}.$$

Now all terms on the right hand side except for $2\frac{\partial H}{\partial t}$ are bounded. In particular, if two manifolds M_1, M_2 meet along Σ in such a way that $H_1 < -H_2$, then the last term $\frac{\partial H}{\partial t}$ behaves like a positive Dirac-Delta function in M supported on Σ.

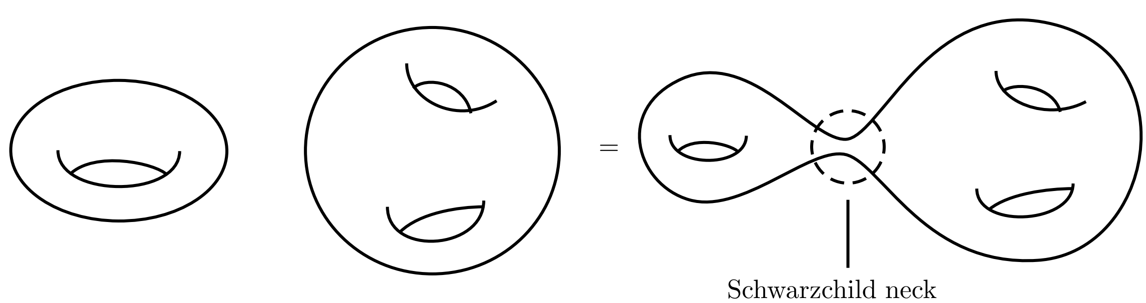

FIGURE 3. Connect sum two manifold with a Schwarzschild neck

Using this gluing-smoothening techinque, we will construct a metric in the unit ball with negative total scalar curvature. The basic model we use in our construction are Delaunay surfaces. It is a periodic constant mean curvature surface in $\mathbb{R}^3$. Picture 4 is a picture [1] of this surface.

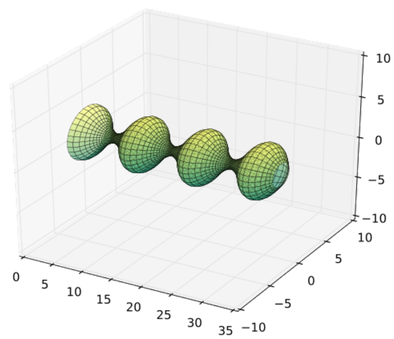

FIGURE 4. A Delaunay surface with constant mean curvature

Take a half ball of radius $\frac{1}{2}$. Dig out a portion of the Delaunay surface described above. Of course in this procedure the constant mean curvature property is no longer preserved. However the total negative part of the scalar curvature can be made as small as possible. Therefore we may cap off a suffiently large part of the Delaunay surface to make it has positive total mean curvature from outside, or negative total mean curvature from inside, as illustrated by the picture. Dig out sufficiently many such 'Delaunay strings' $S_1, \ldots, S_k$, we have that the total mean curvature of the boundary of the manifold $B'_{1/2} = B_{1/2} - S_1 - \ldots - S_k$ is less than -1. Now take two copies of $B'_{1/2}$, reflectively symmetric across the unit disk, and identify the interior boundary from the Delaunay strings. By doing so we get a new manifold $\tilde{B}_{1/2}$ which is diffeomorphic to a ball with a metric C^0 across the boundary. By the gluing-smoothening technique discussed above we may smoothen its metric, and since now the total mean curvature along the boundary is sufficiently negative, the total scalar curvature after smoothening is negative. Denote this new Riemannian manifold $(B_{1/2}, g)$. Take $B_{1/2} \subset B_1$ and extend the metric g trivially into a new metric g in B_1.

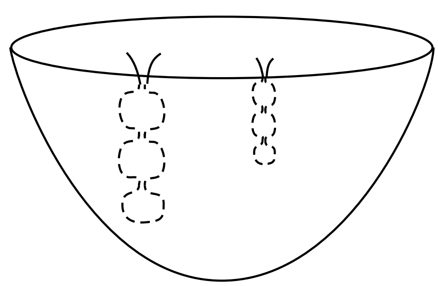

FIGURE 5. Digging Delaunay holes in half ball

Consider the Dirichlet problem of $-L$ in B_1. The above shows that the first Dirichlet eigenvalue $\lambda_1(-L) < 0$. Take the first eigenfunction u and some $\epsilon > 0$ such that $u > \epsilon$ in $B_{1/2}$. Define $v = \frac{1}{\epsilon}\max\{u, \epsilon\}$ and smoothen it in B_1 such that it is still super harmonic. With a slight abuse of notation we still call this smoothened function v. Therefore we get a function v with the following properties:

$$v \equiv 1 \text{ on } \partial B_1, \quad Lv \geq 0.$$

[1]Picture from https://en.wikipedia.org/wiki/Unduloid

Using $v^{4/n-2}$ as the conformal factor, we see that $R(v^{4/n-2}g) < 0$ in $B_{1/2}$. From above we conclude

PROPOSITION 4.5. *There exits a metric g in B_1 such that g is Euclidean on ∂B_1 but the scalar curvature is negative in $B_{1/2}$.*

Note that the above construction cannot be done for two dimensional surfaces. Since any manifold is locally Euclidean, we deduce that

PROPOSITION 4.6. *Any compact manifold M^n, $n \geq 3$ admits a Riemannian metric with negative scalar curvature.*

In fact, the following stronger result of J. Lohkamp [**Loh99**] says that one can locally push the scalar curvature down by an arbitrary amount.

THEOREM 4.7. *Let $n \geq 3$ and $(M^n, \partial M, g)$ be Riemannian manifold. Let $f \leq R(g)$ in M, $f = R(g)$ on ∂M be a smooth function. Then for any $\epsilon > 0$ there is a metric g_ϵ which is equal to g on ∂M such that $f - \epsilon \leq R(g_\epsilon) \leq f$. Moreover, the metric g_ϵ can be chosen arbitrarily close to g in C^0 topology.*

As we have seen in this section, having negative scalar curvature does not put any topological obstruction on a smooth compact manifold with dimension at least 3.

5. How about $R > 0$?

A natural and important question in geometry is to characterize curvature conditions locally. For instance, a classical result by Alexandrov suggests that positive sectional curvature can be charazterized by the Toponogov theorem on triangles. Recently people have been able to give a good characterization of positive Ricci curvature via optimal transport, see [**LV09**], [**Stu06a**] and [**Stu06b**]. How about positive scalar curvature?

As a motivating example, take any two dimensional Riemannian disk $(\Omega, \partial\Omega)$ with boundary $\partial\Omega$ diffeomorphic to S^1. By the Gauss-Bonnet theorem

$$\int_\Omega K dA = 2\pi - \int_{\partial\Omega} k ds.$$

Embed $\partial\Omega$ as a round S^1 in $\mathbb{R}^2$. Under this embedding, $\int_{\partial\Omega} k_0 ds = 2\pi$. Therefore $\int_\Omega K = \int_{\partial\Omega} k_0 - \int_{\partial\Omega} k$, suggesting that $K \geq 0$ implies $\int_{\partial\Omega} k_0 \geq \int_{\partial\Omega} k$.

For higher dimensional manifolds, we have the following result of Y. Shi and L-F Tam [**ST02**].

THEOREM 5.1. *Suppose $(M^3, \partial M = \Sigma^2)$ has nonnegative scalar curvature. Assume the mean curvature $H_\Sigma > 0$ and the Gauss curvature $K_\Sigma > 0$. Isometrically embed Σ^2 into $\mathbb{R}^3$ with mean curvature function H_0. Then*

$$\int_\Sigma (H_0 - H) dA \geq 0,$$

with equality only when M is flat.

Recently Gromov suggested using polyhedron to characterize positive scalar curvature. His proposal is the following: for a tetrahedron or a cube that has mean convex surfaces in a manifold with nonnegative scalar curvature, the dihedral angles cannot be everywhere less than those of the regular figures in $\mathbb{R}^3$.

CHAPTER 2

The positive mass theorem

1. Manifolds admitting metrics with positive scalar curvature

Previously we have seen that there is no topological or geometric constraints for a smooth closed compact manifold to have negative scalar curvature. In fact by the theorem of Lohkamp we may arbitrarily push down the scalar curvature. On the other hand, deforming the metric to increase the scalar curvature may be hard in general. As a first observation, we have

PROPOSITION 1.1. *Suppose (M^n, g_0) is a compact and closed manifold with $R(g_0) \equiv 0$ and $\mathrm{Ric}(g_0) \neq 0$ at some point. Then there is a nearby metric g such that $R(g) > 0$ everywhere.*

PROOF. The simplest mordern proof is through Ricci flow. Under the Ricci flow

$$\begin{cases} \frac{\partial g}{\partial t} = -2\,\mathrm{Ric}_{g_t} \\ g(0) = g_0 \end{cases}$$

The scalar curvature evolves by

$$\frac{\partial R}{\partial t} = \Delta R + 2|\,\mathrm{Ric}_{g_t}|^2.$$

By the maximum principle for all $t > 0$ such that a smooth solution exists, $R > 0$ everywhere. The result follows by the short time existance of the Ricci flow. $\square$

REMARK 1.2. We may also look at the question from a variational point of view. Recall that under a deformation $g_t = g_0 + th$ the total scalar curvature changes by

$$\begin{aligned} \frac{d}{dt}|_{t=0}\mathscr{R}(g_0 + th) &= -\int_M \left\langle \mathrm{Ric}_{g_0} - \frac{1}{2}R_{g_0}, h \right\rangle d\,\mathrm{Vol}_{g_0} \\ &= -\int_M \langle \mathrm{Ric}_{g_0}, h\rangle\, d\,\mathrm{Vol}_{g_0} \end{aligned}$$

Where we have used that $R_{g_0} \equiv 0$. Note that if Ric_{g_0} is not identically zero, then by choosing $h = -\mathrm{Ric}_{g_0}$ the total scalar curvature will be positive. Of course, having positive total scalar curvature along is not enough to guarantee that the scalar curvature is pointwise positive. However, one may calculate the deformation of the first eigenvalue of the conformal Laplacian to conclude that the scalar curvature can be made everywhere positive.

Let's discuss a little bit the classification of manifolds admitting a metric with positive scalar curvature.

EXAMPLE 1.3. *Let (M^4, g_0) be a K3 surface. That is, a four dimensional closed compact simply connected Calabi-Yau manifold. It is Ricci flat, spin and has nonzero $\hat{A}$ genus. By a Dirac operator argument M does not have any metric with positive scalar curvature.*

We point here that in the study of positive scalar metrics, there are two main approaches. One is through the study of minimal hypersurfaces, the other is through Diric operator on spin manifolds. We will focus on the first approach here. We recommend a nice book of B. Lawson and M. Michelsohn [**LM89**] of the Dirac operator approach for interested readers.

The next example shows the subtlety of deforming the scalar on a Ricci flat manifold.

EXAMPLE 1.4. *Let $n \geq 3$, (M^{2n}, g_0) be a Calabi-Yau manifold. In their work* [**DWW05**] *X. Dai, X. Wang and G. Wei studied the second variation of the total scalar curvature functional and proved that the Calabi-Yau metric g_0 is a local maximum. On the other hand, there exists a metric g which is 'far' from g_0 such that $\mathscr{R}(g) > 0$.*

For simply connected manifold (M^n, g) with $n \geq 5$, there is a complete classification for positive scalar curvature:

THEOREM 1.5 ([**GL80**],[**Sto92**]). *Let M be a simply connected manifold of dimension at least 5. Then M carries a metric of positive scalar curvature if and only if:*

- *M does not admit a spin structure, or*
- *M is a spin manifold with certain topological invariant $\alpha(M)$ vanishes (when the dimension of M is a multiple of 4, $\alpha(M)$ is equivalent to $\hat{A}(M)$).*

See also the surgery result in [**YS79**].

For non-simply connected manifolds the existence of a metric with positive scalar curvature is still open. On one hand, take $n \geq 4$ and an aribtrary closed compact manifold M_1^{n-2}. Then the manifold $M^n = M_1^{n-2} \times S^2$ carries a metric g with positive scalar curvature, since we may take a scaling of the standard metric on S^2 such that the sectional curvature there is arbitrarily large. On the other hand, there are certain instances where we do know the non-existence of metric with positive scalar curvature. Let's mention an open question in this direction.

OPEN QUESTION 1.6. Let M^n, $g \geq 4$ be a $K(\pi, 1)$ manifold, that is, the universal cover of M is contratible. Can M carry a metric with positive scalar curvature?

When $n = 3$ with the help of Ricci flow we have a complete classification of manifolds with positive scalar curvature, regardless of the fundamental group:

THEOREM 1.7. *If a closed compact 3-manifold (M^3, g) has positive scalar curvature then there is a finite cover $\hat{M} \to M$ such that $\hat{M} \simeq S^3 \#(S^1 \times S^2) \# \cdots \#(S^1 \times S^2)$.*

2. Positive mass theorem: first reduction

Inspried by the classification result above, one readily sees that any compact closed 3-manifold with a T^3 prime factor does not support any metric with positive scalar curvature. Extending this philosophy to the greatest generality, our central case is to study the existence of positive scalar metrics on a manifold $M^n = M_1^n \# T^n$, where M_1 is an arbitrary compact closed oriented manifold. The best result until today can be summarized as:

THEOREM 2.1. *Let M_1^n, $n \geq 3$ a compact closed oriented manifold. Then $M^n = M_1^n \# T^n$ does not have any metric g with positive scalar curvature, if either*

- *M_1 is spin manifold, or*
- *$n \leq 8$.*

It is believed that these restrictions are techinical. The first case was done by Witten [**Wit81**] with spinnors and index theory, and a spin structure is necessary for the argument; The second case uses the theory of minimal hypersurfaces, and the dimension restriction is to prevent the singularity of area minimizing hypersurfaces. As will be seen shortly, in higher dimensions the possibility of singularities of area minimizing hypersurfaces adds significant amount of difficulty of carrying out a similar argument. Our first goal is to remove the dimension restriction.

Let's start the discussion of the positive mass theorem. Under the classical setup [1], the objects we consider are asymptotically flat manifolds.

[1]There are versions of the positive mass theorem for manifolds with lower regularity.

Definition 2.2. A complete, noncompact smooth manifold (M^n, g) is called **asymptotically flat** (with one end), if outside a compact subset K, $M - K \simeq \mathbb{R}^n - B_1(0)$. Let $x_1, x_2, \ldots, x_n$ be the pull back of the Euclidean coordinates via this diffeomorphism. We require the metric g to satisfy

$$\begin{aligned} &g_{ij} = \delta_{ij} + O(|x|^{-p}), \quad \text{for some } p > \frac{n-2}{2}, \\ &\partial g_{ij} = O(|x|^{-p-1}), \quad \partial^2 g_{ij} = O(|x|^{-p-2}) \\ &R(g) = O(|x|^{-q}), \quad \text{for some } q > n. \end{aligned}$$

We abbreviate the first decay condition by $g_{ij} = \delta_{ij} + O_2(|x|^{-p-1})$.

Example 2.3 (Schwarzschild). *On $\mathbb{R}^n - \{0\}$ the metric*

$$g_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}, \quad m \geq 0.$$

It is not hard to see that this metric is scalar flat and equals $\delta + O_2(|x|^{-n+2})$ near infinity.

The mass of a gravitational system is not easily defined in the most generality. However, for asymptotically flat spaces- or physically isolated gravitating systems- there is a notion of mass called ADM mass (or energy) that mimic the usual Hamiltonian formalism. See [**ADM59**]. We define

Definition 2.4. The ADM mass (introduced by R. Arnowitt, S. Deser and C.W. Misner) for an asymptotically flat manifold is defined by

$$m = \frac{1}{4(n-1)\omega_{n-1}} \lim_{\sigma \to \infty} \int_{|x|=\sigma} (\partial_i g_{ij} - \partial_j g_{ii})\, \nu^j \, dS,$$

where ν is the outward normal vector field of $|x| = \sigma$. ω_{n-1} is the volume of the unit sphere of dimension $n-1$. The constant is chosen such that the ADM mass is equal to m for Schwarzschild.

Remark 2.5. The limit exists in the above definition. Indeed, by Stokes' theorem,

$$\begin{aligned} \int_{|x|=\sigma_2} (\partial_i g_{ij} - \partial_j g_{ii})\, \nu^j \, dS - \int_{|x|=\sigma_1} (\partial_i g_{ij} - \partial_j g_{ii})\, \nu^j \, dS \\ = \int_{\sigma_1 < |x| < \sigma_2} \left(\sum_{i,j} (\partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii}) \right) d\,\mathrm{Vol}\,. \end{aligned}$$

Therefore if $\sum_{i,j} (\partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii})$ is an integrable function then the above boundary integral on balls of radius σ is a Cauchy sequence and hence has a limit. Recall that the scalar curvature, in local coordinates, is given by

$$R(g) = \sum_{i,j} (\partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii}) + O((g - \delta)\partial^2 g) + O((\partial g)^2).$$

It is assumed that

$$g - \delta \approx |x|^{-p}, \quad \partial^2 g \approx |x|^{-p-2}, \quad \partial g \approx |x|^{-p-1}, \quad R(g) \approx |x|^{-q},$$

and intergrability follows by virtue of $|x|^{-2p-2}, |x|^{-q} \in L^1$.

Our main objective is to prove the positive mass theorem in all dimensions, namely

Theorem 2.6 (Positive mass theorem). *Let (M^n, g) be an asymptotically flat manifold and $R(g) \geq 0$. Then its ADM mass $m \geq 0$, and $m = 0$ if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta)$.*

REMARK 2.7. The above formulation is also called the strong positive mass theorem. The weak PMT is usually referred without the rigidity part. Namely, $m_{ADM} \geq 0$ for asymptotically flat manifolds with nonnegative scalar curvature.

In this section we first provide a connection of the positive mass theorem between the existence of metric with positive scalar curvature on a compact manifold.

THEOREM 2.8 (Compactification theorem). *Suppose for all closed compact manifold M_1^n, $M^n = M_1^n \# T^n$ has no metric with positive scalar curvature. Then the ADM mass is nonnegative for all asymptotically flat manifold with nonnegative scalar curvature.*

The proof will be divided into two steps.

Step 1 If (M, g) is asymptotically flat with $m_{ADM} < 0$ then there is a metric $\tilde{g}$ with conformally flat asymptotics and $\tilde{m}_{ADM} < 0$ and $R(\tilde{g}) \equiv 0$.

DEFINITION 2.9. Call a metric $\tilde{g}$ conformally flat asymptotics, if

$$\begin{cases} \tilde{g}_{ij} = u^{\frac{4}{n-2}}\delta_{ij} & \text{outside a compact set} \\ \Delta u = 0 & \text{near } \infty \\ u \to 1 & \text{as } |x| \to \infty \end{cases}$$

Step 2 This is an observation due to J. Lohkamp. If (M, g) has conformally flat asymptotics and $m_{ADM} < 0$ then there exists a metric $\tilde{g}$ with $R(\tilde{g}) > 0$ and $\tilde{g} = \delta$ near ∞.

Assuming these results, the compactification theorem follows. Take M_1^n to be a manifold diffeomorphic to the 1-point compactification of M. Take a big cube C on $(M^n, \tilde{g})$ such that $\tilde{g}$ is Euclidean outside C. Since a neighborhood of the faces of C are Euclidean, and C is the fundamental domain of the group $\mathbb{Z}^n$, we may isometrically identify the corresponding faces, and get a compact manifold M. As a result the manifold $M_1 \# T^n$ is diffeomorphic to $C/\sim$ and carries a metric with positive scalar curvature.

In the proof we are going to use the following important function property of asymptotically flat manifolds.

PROPOSITION 2.10. *Suppose (M^n, g) is an asymptotically flat manifold. Then there exists a constant $\epsilon_0 = \epsilon_0(g)$, such that if f is a smooth function, $f \in L^q \cap L^{\frac{2n}{n+2}}$ with $q > \frac{n}{2}$ and $\|f_-\|_{L^{n/2}} < \epsilon_0$. Then the equation*

$$\begin{cases} \Delta u - fu = 0 & \text{on } M \\ u \to 1 & \text{as } |x| \to \infty \end{cases} \tag{2.1}$$

has a unique positive solution. Moreover, near infinity u has asymptotics

$$u = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{-2}). \tag{2.2}$$

PROOF. The proof we include here are taken from [**SY79**]. Another nice treatment using weighted Sobolev space can be found in [**Bar86**].

Let $v = 1 - u$. We then try to solve the equation

$$\begin{cases} \Delta v - fv = -f & \text{on } M \\ v \to 0 & \text{as } |x| \to \infty \end{cases} \tag{2.3}$$

On a compact subset B_r, consider the Dirichlet problem

$$\begin{cases} \Delta v_r - fv_r = -f & \text{in } B_r \\ v_r = 0 & \text{on } \partial B_r \end{cases} \tag{2.4}$$

We show that equation 2.4 has a unique solution. By Fredholm alternative, it suffices to prove that the homogeneous equation

$$\begin{cases} \Delta w - fw = 0 \quad \text{in } B_r \\ w = 0 \quad \text{on } \partial B_r \end{cases} \tag{2.5}$$

has no nonzero solution, provided $\|f_-\|_{L^{n/2}}$ is sufficiently small. In fact, suppose w is a solution of equation 2.5. Multiply w on both sides of 2.5 and integrate by parts, we have

$$\int_{B_r} |\nabla w|^2 = -\int_{B_r} fw^2 \le \int_{B_r} f_- w^2. \tag{2.6}$$

By the Cauchy-Schwartz inequality and the Sobolev inequality,

$$\begin{aligned} \int_{B_r} f_- w^2 &\le \left(\int_{B_r} f_-^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_{B_r} w^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\le c_1 \left(\int_{B_r} f_-^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_{B_r} |\nabla w|^2\right). \end{aligned} \tag{2.7}$$

Combine equation 2.6 and 2.7 we see that $w = 0$, provided $\|f_-\|_{L^{n/2}} < 1/c_1$. Hence 2.4 has a unique solution v_r. Multiply v_r on both sides of 2.4, using Cauchy-Schwartz and Sobolev inequality again, we see that

$$\begin{aligned} \int |\nabla v_r|^2 &\le \int f_- v_r^2 + \int f v_r \\ &\le c_1 \left(\int f_-^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int |\nabla v_r|^2\right) + c_1 \left(\int f^{\frac{2n}{n+2}}\right)^{\frac{2n}{n+2}} \left(\int |\nabla v_r|^2\right). \end{aligned} \tag{2.8}$$

Therefore there is a constant c_2 depending on (M, g, f) such that $\|v_r\|_{L^{\frac{2n}{n-2}}} < c_2$ and $\|\nabla v_r\|_{L^2} < c_2$. The standard theory of elliptic equations conclude that v_r has uniformly bounded $C^{2,\alpha}$ norm. By Arzela-Ascoli we may pass to a limit and conclude that 2.3 has a solution.

A similar argument proves that the solution is nonnegative everywhere. Otherwise there exists an open set Ω such that

$$\begin{cases} \Delta u - fu = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This contradicts the Sobolev inequality and the choice of ϵ_0 as above. By the strong maximum principle u is positive everywhere.

To get the asymptotic behavior of the solution, we take the Green's function the Laplace operator on M. To do so consider the function

$$Q(x, y) = \left[\sum_{i,j} g_{ij}(x)(y^i - x^i)(y^j - x^j)\right]^{\frac{n-2}{2}}. \tag{2.9}$$

Then we have, by virtue of asymptotic flatness, that

$$\begin{gathered} c_3^{-1}|x-y|^{n-2} \le Q(x,y) \le c_3|x-y|^{n-2}, \quad c_4^{-1}|x-y|^{n-3} \le \partial_y Q(x,y) \le c_4|x-y|^{n-3}, \\ \lim_{|x|\to\infty} |x|^{n-2} Q(x,y) = 1. \end{gathered} \tag{2.10}$$

The function $Q(x, y)^{-1}$ resembles the Green's function in a precise manner, namely

$$\Delta_y Q(x,y)^{-1} = -(n-1)\omega_{n-1}\delta_x(y) + \xi_x(y), \quad \xi \text{ is a function with rapid decay.} \tag{2.11}$$

Multiply $Q(x,y)^{-1}$ on both sides of 2.3 and integrate on a large region $D_r(x)$, we find that

$$(2.12)\quad \begin{aligned}(n-1)\omega_{n-1}v(x) = \int_{D_r} v(y)\xi_x(y)d\,\mathrm{Vol} - \int (fv+f)(y)Q(x,y)^{-1}d\,\mathrm{Vol} \\ + \int_{\partial D_r}\frac{\partial v}{\partial n}Q(x,y)^{-1}dS - \int_{\partial D_r} v(y)\frac{\partial}{\partial n}Q(x,y)^{-1}dS.\end{aligned}$$

By a direct calculation one checks that every term on the right hand side of 2.12 converges to zero as r approaches to infinity, except for $\int (fv+f)(y)Q(x,y)^{-1}d\,\mathrm{Vol}$. Using 2.10 we find that

$$(2.13)\quad \lim_{|x|\to\infty}|x|^{n-2}\int_M (fv+f)(y)Q(x,y)^{-1}d\,\mathrm{Vol}(y) = \int_M (fv+f)(y)d\,\mathrm{Vol}(y).$$

This proves the desired asymptotics. □

The next lemma follows from a direct calculation. It relates the mass of a metric and its conformal change.

LEMMA 2.11. *Suppose (M^n, g) is an asymptotically flat manifold, $u = 1 + \frac{A}{2|x|^{n-2}} + O(|x|^{-n+1})$. Then the mass of (M, g) and of $(M, u^{\frac{4}{n-2}}g)$ are related by*

$$(2.14)\quad m(u^{\frac{4}{n-2}}g) = m(g) + (n-1)A.$$

REMARK 2.12. In fact, it is proved in [**Bar86**] that if the metric is asymptotically flat and conformally flat, then the leading coefficient A in the expansion of u is equal to $m/(n-1)$.

We are now ready to prove theorem 2.8.

PROOF FOR STEP 1. Solve the equation

$$\begin{cases} Lu = 0 & \text{on } M \\ u \to 1 & \text{as } |x| \to \infty. \end{cases}$$

Then the solution u exists and satisfies

$$(2.15)\quad 0 < u < 1, \quad u = 1 - \frac{A}{|x|^{n-2}} + O(|x|^{-2}),$$

where $A = \frac{1}{(n-1)\omega_{n-1}}\int c(n)Ru \geq 0$.

The metric $g' = u^{\frac{4}{n-2}}g$ then satisfies

$$R(g') \equiv 0, \quad m(g') \leq m(g).$$

Denote $g' = \delta + \alpha$, $\alpha_{ij} = O_2(|x|^{-p})$. Take a cutoff function $\Xi(r)$ compactly supported in $B_{2r}(0)$ and is 1 in $B_r(0)$. Consider the metric $\hat{g}$ defined by

$$\hat{g}_{ij}(x) = \delta_{ij} + \Xi(|x|)\alpha_{ij}$$

Then $\hat{g}_{ij} = \delta_{ij} + O_2(|x|^{-p})$ uniformly in r. Since $R(g') = 0$ and $\alpha_{ij} = O(|x|^{-p})$, for r sufficiently large we have that

$$\|R(\hat{g})_-\|_{L^{n/2}} < \epsilon_0.$$ [2]

We then take the solution of

$$(2.16)\quad \begin{cases} L_{\hat{g}}v = 0 & \text{in } M \\ v \to 1 & \text{as } |x| \to \infty. \end{cases}$$

[2] The Sobolev constant is equivalent to the isoperimetric constant. Therefore for a sequence of metrics converging in C^0 their Sobolev constants are bounded. This means that ϵ_0 can be chosen independently of r.

And let $\tilde{g} = v^{\frac{4}{n-2}}\hat{g}$. We see that the metric $\tilde{g}$ then satisfies

$$\begin{cases} \tilde{g}_{ij} = u^{\frac{4}{n-2}}\delta_{ij} & \text{outside a compact set,} \\ \Delta u = 0 & \text{near } \infty, \\ u \to 1 & \text{as } |x| \to \infty. \end{cases} \tag{2.17}$$

In other words, the metric $\tilde{g}$ is conformally flat asymptotics. By choosing r sufficiently large, the mass of $\tilde{g}$ and $\hat{g}$ can be arbitrarily close. Therefore the mass of $\tilde{g}$ is also negative. □

Proof of step 2. Suppose g is conformally flat asymptotics, $g = u^{\frac{4}{n-2}}\delta$, and $m(g) < 0$. Note that u is not a constant function. We are going to see that there exists a metric $\tilde{g}$ such that $R(\tilde{g}) \geq 0$, $\tilde{g} = \delta$ near ∞.

By virtue of proposition 2.10, we know that

$$u(x) = 1 + \frac{A}{2|x|^{n-2}} + O(|x|^{-n+1}), \quad A < 0. \tag{2.18}$$

Therefore we may find $\epsilon > 0$ and $r > 0$ such that $\max_{|x|=r} u < 1 - \epsilon$.

The function $v = \min\{u, 1 - \frac{\epsilon}{2}\}$ is the minimum of two harmonic functions, hence is super-harmonic. Mollify v to be a smooth function such that it is still super-harmonic and equals to $1 - \frac{\epsilon}{2}$ near infinity. By slight abuse of notation we still denote it by v. Note that since u is not a constant function, the function v is not harmonic for sufficiently large r.

Define a metric

$$\tilde{g} = \begin{cases} g & \text{inside } |x| \leq r \\ v^{\frac{4}{n-2}}\delta & \text{when } |x| \geq r. \end{cases} \tag{2.19}$$

Then we find that $R(\tilde{g})$ is nonnegative everywhere and is not identically zero.

Near infinity,

$$\tilde{g}_{ij} = \left(1 - \frac{\epsilon}{2}\right)^{\frac{4}{n-2}} \delta_{ij}.$$

In a new coordinate system $y^i = (1 - \frac{\epsilon}{2})^{\frac{2}{n-2}} x^i$ the metric is Euclidean near infinity.

□

As a last step in this reduction, let us look at the rigidity case of the positive mass theorem. In fact, the structure of asymptotically flat manifolds impose a strong condition of the manifold near infinity that the weak version of the theorem also implies the rigidity case.

Theorem 2.13. *Suppose on any asymptotically flat manifolds (M^n, g), $n \geq 3$, we have the weak positive mass theorem. Namely if the scalar curvature is everywhere nonnegative then its ADM mass is nonnegative. Then we also have the rigidity case. That is, $m = 0$ only when (M^n, g) is isometric to $(\mathbb{R}^n, \delta)$.*

Proof. The proof proceeds by first showing that $R(g) = 0$, then $\mathrm{Ric}(g) = 0$, then $R_{ijkl}(g) = 0$.

We first prove that the scalar curvature of M is identically zero. For the sake of contradiction let us assume $\sup R(g) > 0$ on M. Solve the equation

$$\begin{cases} \Delta u - c(n)Ru = 0 & \text{on } M, \\ u \to 1 & \text{as } |x| \to \infty. \end{cases} \tag{2.20}$$

Then a unique solution u exists and $0 < u < 1$. Define $\hat{g} = u^{\frac{4}{n-2}}g$. Then $R(\hat{g}) = 0$ and

$$m(\hat{g}) = (n-1)m_0 + m(g), \tag{2.21}$$

where m_0 is the leading order term in the expansion of u:

$$u(x) = 1 + \frac{m_0}{2|x|^{n-2}} + O(|x|^{1-n}). \tag{2.22}$$

The fact that $0 < u < 1$ tells us that $m_0 \leq 0$. However, by Proposition 2.10, m_0 can be obtained by integrating equation 2.20 against the Green's function:

$$m_0 = -C_1(n) \int_M R(g)u < 0. \tag{2.23}$$

Therefore $m(\hat{g}) < 0$, contradiction.

Next we prove that the Ricci curvature of M is identically zero. This is done by calculating the first variation of the mass under a compactly supported deformation of the metric. Let h be a compactly supported $(0, 2)$ tensor and $g_t = g + th$, where g is an asymptotically flat metric with zero scalar curvature. Consider the equation

$$\begin{cases} \Delta_t u_t - c(n)R(g_t)u_t = 0, & \text{on } M, \\ u_t \to 1 \quad \text{as } |x| \to \infty. \end{cases} \tag{2.24}$$

For any h since the scalar curvature depends smoothly on t, we have that $\|R(g_t)\|_{L^{3/2}}$ is sufficiently small. Therefore the equation has a unique positive solution u_t by Proposition 2.10. Define $\tilde{g}_t = u_t^{\frac{4}{n-2}} g_t$. Then $R(\tilde{g}_t) = 0$. Let $m(t)$ denote the mass of the metric $\tilde{g}_t$. Using the asymptotic formula again we see that

$$m(t) = -C_1(n) \int_M R(g_t)u_t d\operatorname{Vol}_{g_t}, \tag{2.25}$$

hence in particular $m(t)$ is C^1 differentiable in t. Taking its first derivative at $t = 0$, and use the facts that $u_0 \equiv 1$, $R(g_0) = 0$, we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} m(t) &= -C_1(n) \int_M \dot{R}(0) d\operatorname{Vol} \\ &= C_1(n) \int_M \langle \operatorname{Ric}_g, h \rangle \, d\operatorname{Vol}. \end{aligned} \tag{2.26}$$

See the calculation in equation 2.1 for more details in the above differentiation. If Ric is not identically zero then take $h = \eta \operatorname{Ric}$, η a cutoff function, we see that

$$\left.\frac{d}{dt}\right|_{t=0} m(t) \neq 0.$$

This means that for some small t, $m(t) < 0$, contradiction.

We then prove that the Riemannian curvature tensor is zero on M. Recall that by Bishop-Gromov volume comparision theorem, since $\operatorname{Ric} \geq 0$ on (M, g), the ratio

$$\frac{\operatorname{Vol}(B_\sigma(p))}{\omega_n \sigma^n}$$

is a monotone decreasing function in σ for any point p. However, since (M, g) is asymptotically flat, we get that

$$\lim_{\sigma \to \infty} \frac{\operatorname{Vol}(B_\sigma(p))}{\omega_n \sigma^n} = 1.$$

On the other hand we also know that $\lim_{\sigma \to 0} \frac{\operatorname{Vol}(B_\sigma(p))}{\omega_n \sigma^n} = 1$. Therefore the volume ratio

$$\frac{\operatorname{Vol}(B_\sigma(p))}{\omega_n \sigma^n} \equiv 1,$$

for any point p and any radius σ. By the rigidity statement of Bishop-Gromov theorem we have that (M^n, g) is isometric to $(\mathbb{R}^n, \delta)$.

□

Let us highlight the conclusion of this section:

No metric on $M_1^n \# T^n$ with positive scalar curvature $\Rightarrow$ Positive mass theorem.

3. Minimal slicing

In this section we study metrics on manifolds $M_1^n \# T^n$, where M_1^n is a closed compact oriented manifold. Our goal is to prove

Goal 3.1. For any closed compact oriented manifold M_1^n, $n \geq 3$, there is no metric with positive scalar curvature on $M_1^n \# T^n$.

The idea is to perform induction on the dimension n by constructing a nested family of minimizing hypersurfaces with precise control on their scalar curvature in each stage. We begin by introducing the general strategy to do this induction.

3.1. General strategy. We first give some intuition of the conctruction we will make. For the sake of simplicity let us for now ignore the regularity issue that will be dealt with eventually. For $k < n$, the object we would like to have is a nested family of oriented submanifolds

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = (M, g)$$

such that each $\Sigma_{j-1} \subset \Sigma_j$ is a least volume hypersurface for some weighted volume function in its homology class. The construction is done through a backward inductive procedure which we will briefly describe now.

Suppose there is a nontrivial $(n-1)$-dimensional integral homology class in (M^n, g). By the general existence theorem in geometric measure theory, take $\Sigma_{n-1} \subset M$ to be a volume minimizing current representing this class. Let us assume that Σ_{n-1} is a properly embedded minimal hypersurface. Then it is also a stable minimal hypersurface, meaning that the second variation of its volume is always nonnegative. Choose $u_{n-1} > 0$ on Σ_{n-1} to be the first eigenfunction of the Jacobi operator determined by the second variation, and define $\rho_{n-1} = u_{n-1}$.

Assume, by induction, that we have already constructed $\Sigma_{j+1} \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = M$, together with the positive functions u_l, ρ_l on Σ_l, $l \geq j+1$. Then Let Σ_j be a volume minimizer in some nontrivial homology class in Σ_{j+1} of the weighted volume functional

$$V_{\rho_{j+1}}(\Sigma_j) = \int_{\Sigma_j} \rho_{j+1} d\mathcal{H}^j,$$

where $d\mathcal{H}^j$ is the j-dimensional Hausdorff measure induced from the ambient metric on (M^n, g). Assuming regularity of Σ_j, choose u_j on Σ_j to be the first eigenfunction of the Jacobi operator, and inductively define $\rho_j = u_j \rho_{j+1}$.

Definition 3.2. We call a nested family of minimal surfaces

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = (M, g)$$

and positive functions u_j, ρ_j a minimal k-slicing, if $\Sigma_j \subset \Sigma_{j+1}$ is a volume minimizer for the weighted volume functional $V_{\rho_{j+1}}$, u_j is the first Jacobi eigenfunction on Σ_j, and $\rho_j = u_j \rho_{j+1}$.

Example 3.3. *A trivial example of a minimal k-slicing can be constructed in $X^k \times T^{n-k}$, equipped with a product metric $g_X + \delta$, where δ is the flat metric on the torus. Indeed, just take the nested family of totally geodesic embeddings*

$$X \subset X \times S^1 \subset \cdots \subset X \times T^{n-k},$$

with all the Jacobi eigenfunctions and weight functions $u_j = \rho_j = 1$.

The geometric significance of a minimal slicing is the folloiwng.

THEOREM 3.4. *If the scalar curvature on M is positive, then for an appropriately chosen minimal k-slicing*

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = (M, g),$$

Σ_l *is Yamabe positive for every $k \leq l \leq n$. In particular, if $k = 2$ then Σ_2 is diffeomorphic to S^2.*

In example 3.3, if we further let $X = S^2$ equipped with a metric with positive Gauss curvature, then the above theorem just says that the trivial minimal 2-slicing on $S^2 \times T^{n-2}$ given by

$$S^2 \subset S^2 \times S^1 \subset \cdots \subset S^2 \times T^{n-2}$$

is such that each $S^2 \times T^j$ is Yamabe positive.

One sees that the existence of a minimal k-slicing depends on the topology of (M, g). To get the first hypersurface one needs a nontrivial integral homology class α_{n-1} in $H_{n-1}(M, \mathbb{Z})$. In general, a $n - j$ dimensional submanifold in M may be viewed as the intersection of $j - 1$ hypersurfaces. Given $n - k$ integral homology classes $\alpha_1, \cdots, \alpha_{n-k}$ in $H_{n-1}(M, \mathbb{Z})$ such that $\alpha_1 \cap \cdots \cap \alpha_{n-k} \neq 0$, we may minimize the weighted volume in the class $\alpha_1 \cap \cdots \cap \alpha_j$ inductively in the construction of a minimal k-slicing. Using Poincaré duality, this is implied by the existence of $n - k$ one forms $\omega_1, \cdots, \omega_{n-k} \in H^1(M, \mathbb{R})$ such that $\omega_1 \wedge \cdots \wedge \omega_{n-k} \neq 0$.

Note that such an assumption is naturally satisfied by the torus T^n, for every $k = 1, \cdots, n$. Moreover, it is satisfied by any manifold M^n which admits a degree one map to T^n, by pulling back those 1-forms on T^n. In particular, the manifold $M^n = M_1^n \# T^n$ in our consideration has the correct topological structure for the construction of a minimal k-slicing. We will give more details later.

3.2. Geometry of second variation. The proof of Theorem 3.4, granted that all the nested submanifolds Σ_j are embedded, relies on the second variation of a stable minimal hypersurface. Suppose $(\Sigma^{n-1}) \subset (M^n, g)$ is a two-sided embedded stable minimal hypersurface with a unit vector field ν. Then $S(\varphi, \varphi) \geq 0$, for any $\varphi \in C_0^1(\Sigma)$, where the quadratic form S is defined to be

$$S(\varphi, \varphi) = \int_\Sigma \left(|\nabla \varphi|^2 - (|A|^2 + \operatorname{Ric}_M(\nu, \nu))\varphi^2 \right). \tag{3.1}$$

To see the connection between the stability and the scalar curvature, let us choose a local frame $e_1, \cdots, e_n$ with $e_n = \nu$. Then

$$R_M = \sum_{i,j=1}^{n} R^M_{ijji} = 2 \operatorname{Ric}_M(e_n, e_n) + \sum_{i,j=1}^{n-1} R^M_{ijji}. \tag{3.2}$$

By the Gauss equation, $R^M_{ijji} = R^\Sigma_{ijji} + h_{ii}h_{jj} - h^2_{ij}$. Plugging this into 3.2, we get

$$\operatorname{Ric}_M(\nu, \nu) = \frac{1}{2}(R_M - R_\Sigma - |A|^2). \tag{3.3}$$

Therefore the second variation may be rewritten as

$$S(\varphi, \varphi) = \int_\Sigma |\nabla \varphi|^2 - \frac{1}{2}(R_M - R_\Sigma + |A|^2)\varphi^2 = -\int_M \varphi J \varphi, \tag{3.4}$$

$J = \Delta_\Sigma + \frac{1}{2}(R_M - R_\Sigma + |A|^2)$ is the Jacobi operator.

We compare two interpretations of the connection between the stability of a minimal hypersurface and the positivity of scalar curvature.

- Conformal interpretation. If Σ is stable and $R_M > 0$ then Σ is Yamabe positive.

  To see this, let us use the similarity of the Jacobi operator from stability and the conformal Laplacian. We see that for any $\varphi \in C_0^1(\Sigma)$,

$$\begin{aligned} 0 &\leq S(\varphi,\varphi) \\ &< \int_\Sigma |\nabla\varphi|^2 + \frac{1}{2}R_\Sigma\varphi^2 \\ &= \frac{1}{2c(n)}\int_\Sigma 2c(n)|\nabla\varphi|^2 + c(n)R_\Sigma\varphi^2 \\ &\leq \frac{1}{2c(n)}\int_\Sigma |\nabla\varphi|^2 + c(n)R_\Sigma\varphi^2. \end{aligned} \tag{3.5}$$

  Note that we have used $c(n) = \frac{n-2}{4(n-1)} < \frac{1}{2}$. We therefore conclude from the variational characterization that $\lambda_1(-L) > 0$, L is the conformal Laplacian.

  Therefore we may conformally deform the metric on Σ with positive scalar curvature, and find a stable minimal hypersurface of it. The induction may be carried.

- Warp product interpretation. Assume $S(\varphi,\varphi) \geq 0$ for any $\varphi \in C_0^1(\Sigma)$, that is, stability of Σ. Then there exists a positive first eigenfunction u of the Jacobi operator. On the manifold $\Sigma \times S^1$ consider the warp product metric $g_\Sigma + u^2dt^2$, where g_Σ is the induced metric on Σ and t is the parameter on S^1. Then the scalar curvature of this warp product can be easily calculated as

$$R(g_\Sigma + u^2dt^2) = -2u^{-1}(\Delta u - \frac{1}{2}R_\Sigma u).$$

  Since $Ju = -\lambda_1 u$, $\lambda_1 \geq 0$, we then conclude that

$$\begin{aligned} R(g_\Sigma + u^2dt^2) &= -2u^{-1}(\Delta u - \frac{1}{2}R_\Sigma u) \\ &\geq -2u^{-1}(-\frac{1}{2}R_M u - \frac{1}{2}|A|^2u) \\ &= R_M + |A|^2. \end{aligned} \tag{3.6}$$

  Therefore if $R_M \geq \kappa$ then $R(g_\Sigma + u^2dt^2) \geq \kappa$.

  To illustrate the inductive procedure, suppose we already have $\Sigma_{n-1} \subset \Sigma_n$ stable. Then by above calculation $R(g_{n-1} + u_{n-1}^2dt_{n-1}^2) \geq \kappa > 0$. For an embedded hypersurface $\Sigma_{n-2} \subset \Sigma_{n-1}$, we may embed $\Sigma_{n-2} \times S^1$ into $\Sigma_{n-1} \times S^1$ by taking identity map on the S^1 factor. Then the volume of $\Sigma_{n-2} \times S^1$ with respect to the warped product metric $g_{n-1} + dt_{n-1}^2$ is given by

$$\mathrm{Vol}(\Sigma_{n-2} \times S^1, g_{n-1} + dt_{n-1}^2) = \int_{\Sigma_{n-2}} u_{n-1}d\mathcal{H}^{n-2}.$$

  We then minimize the volume of $\Sigma_{n-2} \times S^1$ among all the embedded hypersurfaces Σ_{n-2}, namely consider the minimization problem

$$\inf\{\mathrm{Vol}(\Sigma_{n-2} \times S^1, g_{n-1} + dt_{n-1}^2) : \Sigma_{n-2} \subset \Sigma_{n-1} \text{ is an embedded hypersurface}\}. \tag{3.7}$$

  Take $\rho_{n-1} = u_{n-1}$. Note that this is equivalent to the minimization problem

$$\inf\{\mathrm{Vol}(\Sigma_{n-2}, g_{n-1}) = \int_{\Sigma_{n-2}} u_{n-1}d\mathcal{H}^{n-2} : \Sigma_{n-2} \text{ is an embedded hypersurface}\}. \tag{3.8}$$

  In a general inductive procedure, suppose we have already constructed Σ_{n-j+1}. Then for an embedded hypersurface $\Sigma_{n-j} \subset \Sigma_{n-j+1}$, the volume of the embedded hypersurface

$\Sigma_{n-j} \times T^{j-1} \subset \Sigma_{n-j+1} \times T^{j-1}$ is given by

$$\begin{aligned} &\mathrm{Vol}(\Sigma_{n-j} \times T^{j-1}, g_{n-j+1}+u_{n-j+1}^2 dt_{n-j+1}^2 + \cdots + u_{n-1}^2 dt_{n-1}^2) \\ &\qquad = \int_{\Sigma_{n-j}} u_{n-j+1} \cdots u_{n-1} d\mathcal{H}^{n-j}. \end{aligned} \tag{3.9}$$

We then minimize for an embedded hypersurface $\Sigma_{n-j} \subset \Sigma_{n-j+1}$, the volume of $\Sigma_{n-j} \times T^{j-1} \subset \Sigma_{n-j+1} \times T^{j-1}$. Note that this is equivalent to the minimization of $\Sigma_{n-j} \subset \Sigma_{n-j+1}$ with the weighted volume.

When the dimension of M is less than or equal to 7, area minimizing hypersurfaces are smooth and embedded. In this case both the conformal interpretation and the warp product interpretation can be used to prove the nonexistence of a metric on $M_1^n \# T^n$ with positive scalar curvature, and the positive mass theorem follows. Such an argument can be generalized to dimension 8 with some extra work. In fact, in dimension 8, only isolated singularities may occur for a minimizing hypersurface $\Sigma^7 \subset M^8$. They can be perturbed away by the result of N. Smale [**Sma93**]:

THEOREM 3.5 ([**Sma93**]). *Suppose M^8 is a closed compact manifold, $\alpha \in H_7(M, \mathbb{Z})$. Then there is a set of metrics, dense in the C^k topology for any k, such that there exists a $\Sigma(g)$ representing α which is homologically volume minimizing and is regular.*

Using this theorem the positive mass theorem also holds in dimension 8: we just need to deform the ambient metric on a manifold $M_1^8 \# T^8$ such that the scalar curvature is still positive everywhere, and the volume minimizing hypersurface is regular.

However, in general dimension singularities of area minimizing currents may occur. The advantage of the warp product interpretation over the conformal interpretation is two-fold: with the presence of singular metric, there is few control of the first eigenfunction near the singularities; Also in the conformal interpretation, $R_M \geq \kappa$ does not imply $R_\Sigma \geq \kappa$. We therefore use the warp product interpretation in our proof.

To better handle the singular set it helps to work with a quadratic form which is more 'coercive'. To see this, define

$$Q(\varphi, \varphi) = S(\varphi, \varphi) + \int_\Sigma P\varphi^2, \tag{3.10}$$

P is a positive function to be chosen later. If the hypersurface is stable then an eigenfunction u of Q provides a bound of P and u^2 in an integral sense, by virtue of

$$\int_\Sigma Pu^2 \leq Q(u, u) \leq \lambda_1 \int_\Sigma u^2.$$

This is an analytic advantage for us.

On the other hand, this extra term a priori may cause some geometric disadvantage, since for the first eigenfunction u,

$$-Ju + Pu = \lambda_1 u, \quad \lambda_1 \geq 0. \tag{3.11}$$

$$\Rightarrow \Delta u + \frac{1}{2}(R_M - R_\Sigma + |A|^2)u^2 - Pu \leq 0. \tag{3.12}$$

We then see that the new scalar curvature of the warped product metric $g_\Sigma + u^2 dt^2$ becomes

$$R(g + u^2 dt^2) \geq R_M + |A|^2 - P. \tag{3.13}$$

Therefore it is safe to pick $P \sim \frac{1}{2}|A|^2$ to keep the scalar curvature lower bound. The idea is to choose P_j on Σ_j, on one hand large enough such that it makes the analysis work, and on the other hand small enough such that it still makes the scalar curvature uniformly bounded from below.

3.3. Calculation at regular points. We fix some notations and derive necessary geometric formulas that hold at smooth points. Ignoring regularity issues for now, consider a smooth minimal k-slicing

$$\Sigma_k \subset \cdots \subset \Sigma_n,$$

Σ_j minimizes the weighted volume $V_{\rho_{j+1}}(\cdot)$ among oriented hypersurfaces in Σ_{j+1}. Let g_k be the metric on Σ_k induced by the embedding $\Sigma_k \subset \Sigma_n$, ν_k be the unit normal vector field of $\Sigma_k \subset \Sigma_{k+1}$. Then $\nu_k, \cdots, \nu_{n-1}$ forms an orthonormal basis of the normal bundle of Σ_k. Denote A_k the vector valued second fundamental form of Σ_k in Σ_n, that is, for any tangent vectors X, Y on Σ_k,

$$A_k(X,Y) = (\nabla^{(n)}_X Y)^\perp = \sum_{p=k}^{n-1} A_k^{\nu_p}(X,Y)\nu_p,$$

where $A_k^{\nu_p}$ is the scalar valued second fundamental form defined by $A_k^{\nu_p} = \langle A_k, \nu_p\rangle$. Clearly we then have

$$|A_k|^2 = \sum_{p=k}^{n-1} |A_k^{\nu_p}|^2.$$

There are two more relavent metrics $\tilde{g}_j$ and $\hat{g}_j$. On $\Sigma_j \times T^{n-j}$ define a warped product metric

$$\hat{g}_j = g_j + \sum_{p=j}^{n-1} u_p^2 dt_p^2. \tag{3.14}$$

Here $(t_j, \cdots, t_{n-1})$ are variables on T^{n-j}, u_p is the first eigenfunction of a quadratic form Q_p on Σ_p which we will define later. Then

$$\mathrm{Vol}(\Sigma_j \times T^{n-j}, \hat{g}_j) = \int_{\Sigma_j} \rho_j d\nu_j,$$

$d\nu_j$ is just the j dimensional Hausdorff measure.

Recall that $\Sigma_j \subset \Sigma_{j+1}$ is a minimizer of the weighted volume functional $V_{\rho_{j+1}}$, or equivalently

$$\Sigma_j \times T^{n-j-1} \subset \Sigma_{j+1} \times T^{n-j-1}$$

is minimizing for the metric $\hat{g}_{j+1}$. Let $\tilde{g}_j$ be the metric on $\Sigma_j \times T^{n-j-1}$ induced from the embedding $\Sigma_j \times T^{n-j-1} \subset \Sigma_{j+1} \times T^{n-j-1}$. Equivalently

$$\tilde{g}_j = g_j + \sum_{p=j+1}^{n-1} u_p^2 dt_p^2. \tag{3.15}$$

We point out here that despite their appearing similarity, the metrics $\hat{g}_j$ and $\tilde{g}_j$ have very different geometric behaviors. In particular, the metric $\hat{g}_j$ more or less has positive scalar curvature.

Let $\tilde{A}_j$ be the second fundamental form of $\Sigma_j \times T^{n-j-1} \subset \Sigma_{j+1} \times T^{n-j-1}$. We then have

LEMMA 3.6.

$$\tilde{A}_j = A_j^{\nu_j} - \sum_{p=j+1}^{n-1} u_p(\nu_j u_p) dt_p^2, \tag{3.16}$$

$$|\tilde{A}_j|^2 = |A_j^{\nu_j}|^2 + \sum_{p=j+1}^{n-1} (\nu_j \log u_p)^2. \tag{3.17}$$

Proof. On Σ_{j+1} take a local orthonormal frame $e_1, \cdots, e_{j+1}$ such that e_1 is the unit normal vector field of $\Sigma_j \subset \Sigma_{j+1}$. For $p = 1, \cdots, n-j-1$ let $Y_p = \partial_{t_p}$. With a slight abuse of notation let us denote $e_1, \cdots, e_{j+1}$ the vector fields under the embedding $\Sigma_j \times T^{n-j-1} \subset \Sigma_{j+1} \times T^{n-j-1}$. Note that they are still orthonormal with respect to the metric $\hat{g}_{j+1}$, and e_1 is still the unit normal vector field. For notational simplicity let us abbreviate $\hat{g}_{j+1}$ to $\hat{g}$ and g_{j+1} to g.

To calculate the second fundamental form, we first see that if $X_1, X_2 \in \operatorname{span}\{e_2, \cdots, e_{j+1}\}$ then

$$\begin{aligned}\tilde{A}_j(X_1, X_2) &= \frac{1}{2}\hat{g}^{11}\left(X_1\hat{g}(X_2, e_1) + X_2\hat{g}(X_1, e_1) - e_1\hat{g}(X_1, X_2)\right) \\ &= \frac{1}{2}g^{11}\left(X_1 g(X_2, e_1) + X_2 g(X_1, e_1) - e_1 g(X_1, X_2)\right) \\ &= A_j^{\nu_j}(X_1, X_2).\end{aligned} \tag{3.18}$$

We also have the calculation for $\tilde{A}(e_i, Y_p)$ and $\tilde{A}(Y_p, Y_q)$:

$$\tilde{A}_j(e_i, Y_p) = \frac{1}{2}\hat{g}^{11}(e_i\hat{g}(Y_p, e_1) + Y_p\hat{g}(e_i, e_1) - e_1\hat{g}(e_i, Y_p)) = 0. \tag{3.19}$$

$$\begin{aligned}\tilde{A}_j(Y_p, Y_q) &= \frac{1}{2}\hat{g}^1 1(Y_p\hat{g}(Y_q, e_1) + Y_q\hat{g}(Y_p, e_1) - e_1 g(\widehat{Y_p, Y_q})) \\ &= -\frac{1}{2}e_1\hat{g}(Y_p, Y_q) \\ &= -\delta_p^q\frac{1}{2}e_1[(u_p)^2].\end{aligned} \tag{3.20}$$

We therefore get the expression for $\tilde{A}_j$. Take the square norm with respect to the metric $\tilde{g}_j$ we get the desired formula for $|\tilde{A}_j|^2$.

□

For any function φ on Σ_j, it can be viewed as a function on $\Sigma_j \times T^{n-j}$ which does not depend on T^{n-j}. The Dirichlet integral is then equal to

$$\int_{\Sigma_j} |\nabla_j\varphi|^2\rho_{j+1}d\mu_j,$$

and therefore the weighted Laplacian is given by

$$\tilde{\Delta}_j\varphi = \rho_{j+1}^{-1}\operatorname{div}(\rho_{j+1}\nabla\varphi). \tag{3.21}$$

Let us now define the coercive quadratic form Q.

$$\begin{aligned}Q_j(\varphi, \varphi) = S_j(\varphi, \varphi) + \frac{3}{8}\int_{\Sigma_j}\Bigg[|\tilde{A}_j|^2 \\ + \frac{1}{3n}\sum_{p=j+1}^{n}(|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2)\Bigg]\varphi^2\rho_{j+1}d\mu_j.\end{aligned} \tag{3.22}$$

We also assume that $\tilde{A}_n = 0, u_n = 1$. S_j is the second fundamental form with respect to the weighted metric, given by

$$S_j(\varphi, \varphi) = \int_{\Sigma_j}\left[|\nabla_j\varphi|^2 - \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j + |\tilde{A}_j|^2)\varphi^2\right]\rho_{j+1}d\mu_j. \tag{3.23}$$

Note that the quadratic form Q_j is more coercive than the second fundamental form, and hence by choosing u_j to be its first eigenfunction, the scalar curvature of the warped metric tends to decrease. Nevertheless we still have the geometric theorem, Theorem 3.4, namely

Theorem 3.7. *If $R_n \geq \kappa$ and a minimal k-slicing exists. Then:*

- *For $k \leq j \leq n-1$, if Σ_j is a smooth submanifold then it is Yamabe positive.*

- *If $k=2$ then Σ_2 is a union of two spheres with diameter bounded from above by $\frac{2\pi}{\sqrt{\kappa}}$.*
- *If $k=1$ then Σ_1 is an arc with length bounded from above by $\frac{2\pi}{\sqrt{\kappa}}$.*

We prove this theorem through a number of calculational lemmas. We begin from the following basic lemma in local calculation.

LEMMA 3.8. *The scalar curvature of the warped product Riemannian manifold $(\Sigma\times S^1, g+u^2dt^2)$ is given by*

$$R(g+u^2dt^2) = R(g) - 2u^{-1}\Delta_g u. \tag{3.24}$$

PROOF. Choose $x^1,\cdots,x^n$ to be a local coordinate system on Σ normal at one point. Then $\partial_1,\cdots,\partial_n,\partial t/u$ is a local coordinate system on $\Sigma\times S^1$ normal at one poin. The covariant derivatives involving t are given by

$$\nabla_{\partial_i}\partial_t = \frac{2}{u}\partial_t, \quad \nabla_{\partial_t}\partial_t = -2\sum_i(\partial_i u)u\partial_i.$$

The scalar curvature is then given by

$$\begin{aligned}
R(g+u^2dt^2) &= R(g) + 2\sum_{i=1}^n R_{itti} \\
&= R(g) + 2\sum_i \left(\left\langle \nabla_i\nabla_{\partial_t/u}(\partial_t/u),\partial_i\right\rangle - \left\langle \nabla_{\partial_t/u}\nabla_i(\partial_t/u),\partial_i\right\rangle\right) \\
&= R(g) + 2\sum_i \partial_i(-\frac{1}{u}\partial_i u) - 2\sum_i \frac{1}{u^2}(\partial_i u)^2 \\
&= R(g) - 2u^{-1}\Delta_g u.
\end{aligned}$$

□

LEMMA 3.9. *The scalar curvature of the metric $\tilde{g}_j$ is given by*

$$\tilde{R}_j = R_j - 2\sum_{p=j+1}^{n-1} u_p^{-1}\Delta_j u_p - 2\sum_{j+1\le p<q\le n-1}\left\langle \nabla_j \log u_p, \nabla_j \log u_q\right\rangle. \tag{3.25}$$

Or equivalently,

$$\tilde{R}_j = R_j - 4\rho_{j+1}^{-1/2}\Delta_j(\rho_{j+1}^{1/2}) - \sum_{p=j+1}^{n-1}|\nabla_j \log u_p|^2. \tag{3.26}$$

PROOF. We apply 3.24 finitely many times in an inductive manner. For $k=j+1,\cdots,n-1$ let $\bar{g}_k = g_j + \sum_{p=k}^{n-1} u_p^2 dt_p^2$. We prove the formula

$$\bar{R}_k = R_j - 2\sum_{p=k}^{n-1} u_p^{-1}\Delta_j u_p - 2\sum_{k\le p<q\le n-1}\left\langle \nabla_j \log u_p, \nabla_j \log u_q\right\rangle$$

by a finite reverse induction on k. When $k=n-1$ the formula follows directly from 3.24. Now suppose the formula is correct for $\bar{g}_{k+1}$. Note that $\bar{g}_k = \bar{g}_{k+1} + u_k^2 dt_k^2$. We apply 3.24 to obtain

$$\bar{R}_k = \bar{R}_{k+1} - 2u_k^{-1}\bar{\Delta}_{k+1}u_k.$$

Note that u_k does not depend on the extra variables t_p. We use 3.21 to write $\bar{\Delta}_{k+1}$ in terms of Δ_j:

$$u_k^{-1}\bar{\Delta}_{k+1}u_k = u_k^{-1}\rho_{k+1}^{-1}\operatorname{div}_j(\rho_{k+1}\nabla_j u_k) = u_k^{-1}\Delta_j u_k + \sum_{p=k+1}^{n-1}\left\langle \nabla_j \log u_p, \nabla_j \log u_k\right\rangle.$$

Here $\rho_k = u_k\cdots u_{n-1}$. The formula follows from induction.

To prove 3.26, we observe that the cross terms appear in 3.25 can be rewritten as

$$2\sum_{j+1\le p<q\le n-1}\langle\nabla_j\log u_p,\nabla_j\log u_q\rangle=\left|\sum_{p=j+1}^{n-1}\nabla_j\log u_p\right|^2-\sum_{p=j+1}^{n-1}|\nabla_j u_p|^2.$$

Therefore

$$\begin{aligned}\tilde{R}_j&=R_j-2\sum_p\left(\Delta_j\log u_p+|\nabla_j\log u_p|^2\right)-2\sum_{j+1\le p<q\le n-1}\langle\nabla_j\log u_p,\nabla_j\log u_q\rangle\\&=R_j-2\Delta_j\log\rho_{j+1}-\sum_p|\nabla_j\log u_p|^2-|\nabla_j\log\rho_{j+1}|^2\\&=R_j-4\rho_{j+1}^{-1/2}\Delta_j(\rho_{j+1}^{1/2})-\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2.\end{aligned}$$

□

Next we calculate the scalar curvature of the metric $\hat{g}_j$. We point out again that $\hat{g}_j$ and $\tilde{g}_j$ are very different geometrically.

LEMMA 3.10. *The scalar curvature of the metric $\hat{g}_j$ is given by*

$$\hat{R}_j=R_n+2\sum_{p=j}^{n-1}\lambda_p+\frac{1}{4}\sum_{p=j}^{n-1}\left[|\tilde{A}_p|^2-\frac{1}{n}\sum_{q=p+1}^{n-1}(|\nabla_p\log u_q|^2+|\tilde{A}_q|^2)\right],\tag{3.27}$$

where λ_p is the first eigenvalue of the quadratic form Q_p.

PROOF. Recall that u_k is the first eigenfunction of Q_k associated to λ_k, and that Q_k is given by

$$\begin{aligned}Q_k(\varphi,\varphi)=\int_{\Sigma_k}\Bigg[&|\nabla_k\varphi|^2-\frac{1}{2}(\hat{R}_{k+1}-\tilde{R}_k)\varphi^2\\&-\frac{1}{8}\left(|\tilde{A}_k|^2-\frac{1}{n}\sum_{p=k+1}^{n}(|\nabla_k\log u_k|^2+|\tilde{A}_k|^2)\right)\varphi^2\Bigg]\rho_{k+1}d\mu_k.\end{aligned}$$

Denote L_k the linear operator associated to Q_k. Then

$$L_k=\tilde{\Delta}_k+\frac{1}{2}(\hat{R}_{k+1}-\tilde{R}_k)+\frac{1}{8}\left(|\tilde{A}_k|^2-\frac{1}{n}\sum_{p=k+1}^{n}(|\nabla_k\log u_p|^2+|\tilde{A}_p|^2)\right),$$

and u_p satisfies the eigenfunction equation $L_ku_k=-\lambda_ku_k$, $\lambda_k>0$.

We prove 3.27 by a reverse induction beginning with $j=n-1$. From 3.24 we have that $\hat{R}_{n-1}=R_{n-1}-2u_{n-1}^{-1}\Delta_{n-1}u_{n-1}$. The equation u_{n-1} satisfies is

$$\Delta_{n-1}u_{n-1}+\frac{1}{2}(R_n-R_{n-1})u_{n-1}+\frac{1}{8}|\tilde{A}_{n-1}|^2u_{n-1}=-\lambda_{n-1}u_{n-1},$$

and so we have $\hat{R}_{n-1}=R_n+2\lambda_{n-1}+\frac{1}{4}|\tilde{A}_{n-1}|^2$. This proves the result for $j=n-1$.

Suppose 3.27 is satisfied by $\hat{g}_{j+1}$. We first observe that $\hat{g}_j = \tilde{g}_j + u_j^2 dt_j^2$, and therefore $\hat{R}_j = \tilde{R}_j - 2u_j^{-1}\tilde{\Delta}_j u_j$. On the other hand u_j satisfies the equation

$$\begin{aligned}&\tilde{\Delta}_j u_j + \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j)u_j \\ &\quad + \frac{1}{8}\left(|\tilde{A}_j|^2 - \frac{1}{n}\sum_{p=j+1}^{n}(|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2)\right)u_j = -\lambda_j u_j.\end{aligned}$$

Substituting this above we have that

$$\begin{aligned}\hat{R}_j &= \tilde{R}_j + 2\Bigg[\lambda_j + \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j) \\ &\quad + \frac{1}{8}\left(|\tilde{A}_j|^2 - \frac{1}{n}\sum_{p=j+1}^{n}(|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2)\right)\Bigg] \\ &= 2\lambda_j + \hat{R}_{j+1} + \frac{1}{4}\left(|\tilde{A}_j|^2 - \frac{1}{n}\sum_{p=j+1}^{n}(|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2)\right).\end{aligned}$$

Therefore 3.27 follows from the induction. □

LEMMA 3.11. *Assume the scalar curvature $R_n \geq \kappa$. Then $\hat{R}_j \geq \kappa - \frac{1}{4}\sum_{p=j}^{n-1}|\nabla_j \log u_p|^2$.*

PROOF. From 3.17 we have the expression for the second fundamental form $|\tilde{A}_j|^2$. Therefore

$$\begin{aligned}&\sum_{p=j}^{n-1}\left(n|\tilde{A}_p|^2 - \sum_{q=p+1}^{n-1}(|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2)\right) \\ &\qquad \geq \sum_{p=j}^{n-1}\left(\sum_{r=j}^{n}|\tilde{A}_r|^2 - \sum_{q=p+1}^{n}(|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2)\right) \\ &\qquad \geq \sum_{p=j}^{n-1}\sum_{q=p+1}^{n}\left[\sum_{r=j}^{p-1}(\nu_r \log u_q)^2 - |\nabla_p \log u_q|^2\right] \\ &\qquad = -\sum_{p=j}^{n-1}\sum_{q=p+1}^{n}|\nabla_{j-1}\log u_q|^2 \\ &\qquad \geq -n\sum_{q=j}^{n}|\nabla_j \log u_q|^2.\end{aligned}$$

We therefore conclude that if $R_n \geq \kappa$ then $\hat{R}_j \geq \kappa - \frac{1}{4}\sum_{p=j}^{n-1}|\nabla_j \log u_p|^2$. □

Combe this with 3.17, we find that

$$\begin{aligned}|\tilde{A}_j|^2 + \hat{R}_{j+1} &\geq \kappa + \sum_{p=j+1}^{n-1}(\nu_j \log u_p)^2 - \frac{1}{4}\sum_{p=j+1}^{n-1}|\nabla_{j+1}\log u_p|^2 \\ &\geq \kappa - \frac{1}{4}\sum_{p=j+1}^{n-1}|\nabla_j \log u_p|^2.\end{aligned} \tag{3.28}$$

The importance of 3.28 is that it implies the following **unweighted estimates** using stability.

Proposition 3.12. *Assume stability ($S_j(\varphi,\varphi)\geq 0$) on Σ_j equipped with the warped metric, that is, for any smooth function φ,*

$$\int_{\Sigma_j}(\hat{R}_{j+1}+|\tilde{A}_j|^2-\tilde{R}_j)\varphi^2\rho_{j+1}d\mu_j\leq 2\int_{\Sigma_j}|\nabla_j\varphi|^2\rho_{j+1}d\mu_j. \tag{3.29}$$

Then we have, for any smooth function φ,

$$\int_{\Sigma_j}\left(\kappa+\frac{3}{4}\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2-R_j\right)\varphi^2 d\mu_j\leq 4\int_{\Sigma_j}|\nabla_j\varphi|^2 d\mu_j. \tag{3.30}$$

Proof. We use 3.28 and 3.26 into the stability inequality 3.29 and get that, for any smooth function φ,

$$\begin{aligned}&\int_{\Sigma_j}\left[\kappa-\frac{1}{4}\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2-R_j+4\rho_{j+1}^{-1/2}\Delta_j\rho_{j+1}^{1/2}+\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2\right]\varphi^2\rho_{j+1}d\mu_j\\ &\leq 2\int_{\Sigma_j}|\nabla_j\varphi|^2 d\mu_j\\ &\leq 4\int_{\Sigma_j}|\nabla_j\varphi|^2 d\mu_j.\end{aligned} \tag{3.31}$$

Replacing φ by $\varphi\rho_{j+1}^{-1/2}$, we are able to cancel the weight ρ_{j+1} on the left hand side. For the right hand side we see that

$$4\int|\nabla(\varphi\rho_{j+1}^{-1/2})|^2\rho_{j+1}=4\int|\nabla\varphi|^2+2\varphi\rho_{j+1}\nabla\varphi\cdot\nabla(\rho_{j+1}^{-1/2})+\varphi^2|\nabla_j\rho_{j+1}^{-1/2}|^2\rho_{j+1}.$$

Using integration by parts, we have

$$\begin{aligned}4\int 2\varphi\rho_{j+1}^{1/2}\nabla\varphi\cdot\nabla\rho_{j+1}^{-1/2}&=-4\int\nabla(\varphi^2)\cdot\frac{\nabla\rho_{j+1}^{1/2}}{\rho_{j+1}^{1/2}}\\ &=4\int\varphi^2\frac{\Delta(\rho_{j+1}^{1/2})}{\rho_{j+1}^{1/2}}-\int\varphi^2|\nabla\log\rho_{j+1}^{1/2}|^2.\end{aligned}$$

Plugging this into 3.31 we see that all terms involving ρ_{j+1} cancel, and the desired estimate follows. □

Now the geometric theorem, Theorem 3.7, is a direct consequence of Proposition 3.12.

Proof of Theorem 3.7. A smooth compact closed manifold Σ^n is Yamabe positive if and only if

$$\int_\Sigma -R\varphi^2\leq c(n)^{-1}\int_\Sigma|\nabla_k\varphi|^2,$$

for any smooth function φ. Since $c(n)=\frac{n-2}{4(n-1)}<\frac{1}{4}$, $c(n)^{-1}>4$. The statement follows from Proposition 3.12.

For the diameter bound, consider any curve Σ_1. Take s to be the arclength parameter, $0\leq s\leq l$. From Proposition 3.12 we have that

$$\kappa\int_0^l\varphi^2 ds\leq 4\int_0^l(\varphi'(s))^2 ds,$$

for any compactly supported smooth function φ. Therefore $\frac{\pi^2}{l^2}\geq\frac{\kappa}{4}$. Therefore $l\leq\frac{2\pi}{\sqrt{\kappa}}$. □

REMARK 3.13. About the sharpness of the constant in the diameter bound, one might speculate that the following example should be optimal. Take $M = S^2 \times T^{n-2}$, the product of a two-sphere and a flat torus, equipped witht the product metric. Then $R = 2K_{S^2} = \kappa$. Then $K_{S^2} = \kappa/2$. By the Bonnet theorem we see that

$$\mathrm{diam}(S^2) \leq \frac{\pi}{\sqrt{\kappa/2}} = \frac{\sqrt{2}\pi}{\sqrt{\kappa}}.$$

In our bound the constant is 2 instead of $\sqrt{2}$. The sharp bound is not known, but when we used the inequality $2 < 4$ we lost information. If the extra term could be exploited the bound might be improved.

3.4. Existence and regularity. We discuss the existence and regularity of a minimal k-slicing. For a volume minimizing hypersurface $\Sigma_{n-1} \subset \Sigma_n$ let $\mathcal{S}_{n-1}$ be the closed subset of singular points, and $\mathcal{R}_{n-1} = \Sigma_{n-1} \setminus \mathcal{S}_{n-1}$ be the regular set. The standard theory of volume minimizing hypersurfaces then implies that $\dim \mathcal{S}_{n-1} \leq n - 8$. For a nested family of currents

$$\Sigma_j \subset \cdots \subset \Sigma_n,$$

we define

DEFINITION 3.14. The regular set $\mathcal{R}_j$ is defined to be the set of points

$$\{x \in \Sigma_j : \text{there is an open neighborhood } O \text{ of } x \text{ such that } O \cap \Sigma_p \text{ is regular, } p = j, \cdots, n-1.\}$$

The singular set $\mathcal{S}_j = \Sigma_j \setminus \mathcal{R}_j$.

We clearly have that

$$\dim(\mathcal{S}_j) \leq \max\{\dim(\mathcal{S}_{j+1}), n-7\}. \tag{3.32}$$

In the regularity theory we are going to prove that

$$\dim(\mathcal{S}_j) \leq j - 3.$$

The basic strategy for this partial regularity theorem is to use a dimension reduction argument and study homogeneous minimal slicings in the Euclidean spaces. As a first step, it is essential to understand the construction of the weight functions ρ_p in presence of singularities. In particular, we will need an argument to prove that each eigenfunction u_p is not concentrated near the singular sets. To do so let us first embed Σ_n into some $\mathbb{R}^N$. For each j and an open set $\Omega \subset \mathbb{R}^N$, define the weighted norms

$$\|\varphi\|^2_{0,j,\Omega} = \int_{\Sigma_j \cap \Omega} \varphi^2 \rho_{j+1} d\mu_j, \tag{3.33}$$

$$\|\varphi\|^2_{1,j,\Omega} = \|\varphi\|_{0,j,\Omega} + \int_{\Sigma_j \cap \Omega} (|\nabla_j \varphi|^2 + P_j \varphi^2) \rho_{j+1} d\mu_j. \tag{3.34}$$

Here P_j is defined by

$$P_j = |A_j|^2 + \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2. \tag{3.35}$$

The term P_j is used to make the weighted norm more coercive, which will give us some analytic advantage in controlling the singular sets. We define the weighted L^2 space and Sobolev spaces $\mathcal{H}_j$, $\mathcal{H}_{j,0}$ as follows.

DEFINITION 3.15. Let $L^2(\Sigma_j,\Omega)$ be the Hilbert spaces of square integrable functions on Σ_j with respect to the measure $\rho_{j+1}\mu_j$.

Let $\mathcal{H}_j, \mathcal{H}_{j,0}$ be the Hilbert space which is the completion with respect to the norm $\|\cdot\|_{1,j}^2$ of functions in $C_0^1(\bar{\Omega}\cap\mathcal{R}_j)$, $C_0^1(\Omega\cap\mathcal{R}_j)$, respectively.

As in the usual definition of Sobolev spaces, the only difference between $\mathcal{H}_j$ and $\mathcal{H}_{j,0}$ is that $\mathcal{H}_{j,0}$ contains those functions in $\mathcal{H}_j$ with zero boundary data on $\Sigma_j\cap\partial\Omega$.

REMARK 3.16. We will say that a minimal k-slicing in an open set Ω is partially regular if $\dim(\mathcal{S}_j)\le j-3$ for $j=k,\cdots,n-1$. It then follows from 3.32 that if the $(k+1)$-slicing associated to the minimal k-slicing is partially regular then $\dim(\mathcal{S}_k)\le\min\{\dim(\mathcal{S}_{k+1}),n-7\}\le k-2$. In the inductive procedure, we therefore assume in the study of u_j that the singular set $\mathcal{S}_j$ is of at least codimension two. Intuitively $\mathcal{S}_j$ has zero capacity hence do not affect the Dirichlet integral. We will prove this with the weighted volume.

Next we prove the existence of u_j, the first eigenfunction of the quadratic form Q_j. To do so we will need a coercivity estimate.

PROPOSITION 3.17. *Suppose $\varphi\in\mathcal{H}_{j,0}(\Omega)$. Then there exists a constant c that only depends on the minimal slicing but not on the choice of φ, such that*

$$c\|\varphi\|_{1,j,\Omega}^2\le Q_j(\varphi,\varphi)+\|\varphi\|_{0,j,\Omega}^2,$$

where Q_j is the quadratic form defined as 3.22.

PROOF. Recall that

$$Q_j(\varphi,\varphi)=S_j(\varphi,\varphi)+\frac{3}{8}\int_{\Sigma_j}\left[|\tilde{A}_j|^2+\frac{1}{3n}\sum_{p=j+1}^{n-1}(|\tilde{A}_p|^2+|\nabla_j\log u_p|^2)\right]\varphi^2\rho_{j+1}d\mu_j,$$

and S_j is the form from the second variation for the weighted volume functional:

$$S_j(\varphi,\varphi)=\int_{\Sigma_j}\left[|\nabla_j\varphi|^2-\frac{1}{2}(\hat{R}_{j+1}-\tilde{R}_j+|\tilde{A}_j|^2)\varphi^2\right]\rho_{j+1}d\mu_j.$$

Denote

$$q_j=\frac{1}{2}(\hat{R}_{j+1}-\tilde{R}_j+|\tilde{A}_j|^2).$$

To prove the desired estimate, it suffices to bound $\|\varphi\|_{1,j,\Omega}^2$ for any C^1 function compactly supported on $\Omega\cap\mathcal{R}_j$, that is, to bound

$$\int_{\Sigma_j\cap\Omega}\varphi^2\rho_{j+1}d\mu_j+\int_{\Sigma_j\cap\Omega}|\nabla_j\varphi|^2\rho_{j+1}d\mu_j+P_j\varphi^2\rho_{j+1}d\mu_j. \tag{3.36}$$

The first term can be handled by $\|\varphi\|_{0,j,\Omega}^2$ by definition. The second term, namely the gradient term, appeared in $S_j(\varphi,\varphi)$, and thus can be controlled:

$$\int_{\Sigma_j\cap\Omega}|\nabla_j\varphi|^2\rho_{j+1}d\mu_j=S_j(\varphi,\varphi)+\int_{\Sigma_j\cap\Omega}q_j\varphi^2\rho_{j+1}d\mu_j.$$

To deal with the third term, we need to treat the quadratic form Q_j more carefully. Denote

$$\bar{P}_j=\frac{3}{8}\left(|\tilde{A}_j|^2+\frac{1}{3n}\sum_{p=j+1}^{n}(|\tilde{A}_p|^2+|\nabla_j\log u_p|^2)\right).$$

We then have the following easy observation:

$$\frac{1}{8n}\left(\sum_{q=j}^{n}|\tilde{A}_q|^2+\sum_{p=j+1}^{n}|\nabla_j\log u_p|^2\right)\le\bar{P}_j\le\frac{3}{8}\left(\sum_{q=j}^{n}|\tilde{A}_q|^2+\sum_{p=j+1}^{n}|\nabla_j\log u_p|^2\right).\tag{3.37}$$

Using 3.17 we may simplify the above expression further:

CLAIM 3.18.

$$\begin{aligned}&\sum_{q=j}^{n-1}|\tilde{A}_q|^2+\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2\\&\qquad=\sum_{q=j}^{n-1}|A_p^{\nu_p}|^2+\sum_{p=j+1}^{n-1}|\nabla_p\log u_p|^2.\end{aligned}\tag{3.38}$$

PROOF OF CLAIM. We have

$$\begin{aligned}\sum_{q=j}^{n-1}|\tilde{A}_q|^2&=\sum_{q=j}^{n-1}|A_q^{\nu_q}|^2+\sum_{q=j}^{n-1}\sum_{p=q+1}^{n-1}(\nu_q\log u_p)^2\\&=\sum_{q=j}^{n-1}|A_q^{\nu_q}|^2+\sum_{p=j+1}^{n-1}\sum_{q=j}^{p-1}(\nu_q\log u_p)^2.\end{aligned}\tag{3.39}$$

Therefore

$$\begin{aligned}\sum_{q=j}^{n-1}|\tilde{A}_q|^2+\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2&=\sum_{q=j}^{n-1}|A_q^{\nu_q}|^2+\sum_{p=j+1}^{n-1}\left(|\nabla_j\log u_p|^2+\sum_{q=j}^{p-1}(\nu_q\log u_p)^2\right)\\&=\sum_{p=j}^{n-1}|A_p^{\nu_p}|^2+\sum_{p=j+1}^{n-1}|\nabla_p\log u_p|^2.\end{aligned}\tag{3.40}$$

Thus the claim is proved. □

Now let us use the geometric relation $|A_p^{\nu_p}|\ge|A_j^{\nu_p}|$, for $p\ge j$. Then by 3.17,

$$\sum_{p=j}^{n-1}|\tilde{A}_p|^2\ge\sum_{p=j}^{n-1}|A_p^{\nu_p}|^2\ge\sum_{p=j}^{n-1}|A_j^{\nu_p}|^2=|A_j|^2.$$

Combine the above estimates with the stability inequality $S_j(\varphi,\varphi)\ge 0$, we have

$$\begin{aligned}\int_{\Sigma_j}P_j\varphi^2\rho_{j+1}d\mu_j&=\int_{\Sigma_j}\left(|A_j|^2+\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2\right)\varphi^2\rho_{j+1}d\mu_j\\&\le\int_{\Sigma_j}\left(\sum_{p=j}^{n-1}|\tilde{A}_p|^2+\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2\right)\varphi^2\rho_{j+1}d\mu_j\\&\le 8nQ(\varphi,\varphi).\end{aligned}$$

A last term we need to bound is $\int_{\Sigma_j}\frac{1}{2}q_j\varphi^2\rho_{j+1}d\mu_j$. By Lemma 3.10, we have

$$\hat{R}_j\le c+\frac{1}{4}\sum_{p=j}^{n-1}|\tilde{A}_p|^2.\tag{3.41}$$

Therefore

$$q_j \le c + \frac{1}{2}\sum_{p=j}^{n-1} |\tilde{A}_p|^2 - \frac{1}{2}\tilde{R}_k.$$

Here c is an upper bound for $R_n + \sum_p \lambda_p$.

Now from Lemma 3.9 we have that

$$-\frac{1}{2}\tilde{R}_j \le \frac{1}{2}|R_j| + \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 + \mathrm{div}_j(X_j),$$

where $X_j = \sum_{p=j+1}^{n-1} \nabla_j \log u_p$ is a vector field. We use the Gauss equation on the regular part of $\Sigma_j \subset \Sigma_n$ and get that

$$|R_j| \le c(1 + |A_j|^2).$$

Therefore

$$q_j \le c + c\sum_{p=j}^{n-1} |\tilde{A}_p|^2 + \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 + \mathrm{div}_j(X_j).$$

We therefore conclude that

$$\int_{\Sigma_j} \left(|\nabla_j \varphi|^2 + \frac{1}{8n} P_j \varphi^2 \right) \rho_{j+1} d\mu_j \le 2Q_j(\varphi, \varphi) + \int_{\Sigma_j} q_j \varphi^2 \rho_{j+1} d\mu_j, \tag{3.42}$$

and

$$\begin{aligned} \int_{\Sigma_j} q_j \varphi^2 \rho_{j+1} d\mu_j \le c \int_{\Sigma_j} &\left(1 + \sum_{p=j}^{n-1} |\tilde{A}_p|^2 + \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 \right) \varphi^2 \rho_{j+1} d\mu_j \\ &+ \int_{\Sigma_j} \mathrm{div}_j(X_j) \varphi^2 \rho_{j+1} d\mu_j. \end{aligned} \tag{3.43}$$

All the terms on the right hand side except for the last one can be bounded by a multiple of $Q_j(\varphi, \varphi)$. The last term can be dealt with using integration by parts. Since φ vanishes on the boundary of $\Omega \cap \Sigma_j$, we have

$$\begin{aligned} \int_{\Sigma_j} \mathrm{div}_j(X_j) \varphi^2 \rho_{j+1} d\mu_j &= -\int_{\Sigma_j} \left\langle X_j, \nabla_j(\varphi^2 \rho_{j+1}) \right\rangle d\mu_j \\ &= -\int_{\Sigma_j} \left\langle X_j, 2\varphi \nabla_j \varphi \right\rangle \rho_{j+1} d\mu_j - \int_{\Sigma_j} \left\langle X_j, \nabla \rho_{j+1} \right\rangle \varphi^2 d\mu_j. \end{aligned} \tag{3.44}$$

By Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left| \int_{\Sigma_j} \left\langle X_j, 2\varphi \nabla_j \varphi \right\rangle \rho_{j+1} d\mu_j \right| &\le \frac{1}{2} \int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} d\mu_j + 2 \int_{\Sigma_j} |X_j|^2 \varphi^2 \rho_{j+1} d\mu_j \\ &\le \frac{1}{2} \int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} d\mu_j + 2 \int_{\Sigma_j} P_j \varphi^2 \rho_{j+1} d\mu_j \\ &\le \frac{1}{2} \int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} d\mu_j + 16n Q_j(\varphi, \varphi). \end{aligned}$$

And similarly,

$$\begin{aligned}&\left|\int_{\Sigma_j}\langle X_j,\nabla\rho_{j+1}\rangle\,\varphi^2 d\mu_j\right|\\&=\left|\int_{\Sigma_j}\langle X_j,\nabla\log\rho_{j+1}\rangle\,\varphi^2\rho_{j+1}d\mu_j\right|\\&=\int_{\Sigma_j}|\nabla_j\log\rho_{j+1}|^2\varphi^2\rho_{j+1}d\mu_j\\&\le\int_{\Sigma_j}P_j\varphi^2\rho_{j+1}d\mu_j\\&\le 8nQ_j(\varphi,\varphi).\end{aligned}$$

We therefore conclude from 3.42 that

$$\frac{1}{8n}\|\varphi\|_{1,j,\Omega}\le 2Q_j(\varphi,\varphi)+(c+24n)Q_j(\varphi,\varphi)+c\|\varphi\|^2_{0,j,\Omega},$$

where c is an upper bound for the scalar curvature of Σ_n and the first eigenvalues λ_p, as desired. □

An essential consequence of this proposition is the non-concentration result for functions in $\mathcal{H}_{j,0}(\Omega)$. By Cauchy-Schwartz inequality we have

$$|H_j|^2\le j|A_j|^2\le jP_j.$$

As a result, we have that

$$\int_{\Sigma_j\cap\Omega}\left(|\nabla_j(\varphi\sqrt{\rho_{j+1}})|^2+|H_j|^2\varphi^2\rho_{j+1}\right)d\mu_j\le 2j\|\varphi\|^2_{1,j,\Omega}. \tag{3.45}$$

Motivated by the classical Michael-Simon Sobolev inequality, a typical term like the left hand side of 3.45 controls the $L^{\frac{2n}{n-2}}$ norm of the function $\varphi\sqrt{\rho_{j+1}}$. In particular, the L^2 norm of the function $\varphi\sqrt{\rho_{j+1}}$ cannot be concentrated on any closed set of Hausdorff dimension less than n. In our context we cannot directly apply the classical Michael-Simon Sobolev inequality because of the weight ρ_{j+1}. Nevertheless we could adapt the idea and prove the non-concentration property directly.

Proposition 3.19 (L^2 non-concentration). *Let $\mathcal{S}$ be a closed set of zero $(j-1)$ Hausdorff measure. Let Σ_j be a member of a partially regular minimal j-slicing in Ω_1. Then for any $\eta>0$ there exists an open neighborhood $V\subset\Omega_1$ containing $V\cap\Omega$ such that*

$$\int_{\Sigma_j\cap V}\varphi^2\rho_{j+1}d\mu_j\le\eta\|\varphi\|^2_{1,j,\Omega},$$

for all functions $\varphi\in\mathcal{H}_{j,0}(\Omega)$.

Proof. From 3.45 we have already seen that

$$\int_{\Sigma_j\cap\Omega}(|\nabla_j(\varphi^2\rho_{j+1})|+|H_j|^2\varphi^2\rho_{j+1})d\mu_j\le c\|\varphi\|^2_{1,j,\Omega}.$$

Therefore it suffices to prove that for any $\eta>0$ there is a neighborhood V such that

$$\int_{\Sigma_j\cap V}\varphi^2\rho_{j+1}d\mu_j\le\eta\left(\int_{\Sigma_j\cap\Omega}(|\nabla_j(\varphi^2\rho_{j+1})|+|H_j|\varphi^2\rho_{j+1})d\mu_j+\int_{\Sigma_j\cap\Omega}\varphi^2\rho_{j+1}d\mu_j\right).$$

For a C^1 vector field X defined in Ω, the first variation formula of minimal hypersurfaces gives

$$\int_{\Sigma_j} \mathrm{div}_{\Sigma_j}(X) d\mu_j = -\int_{\Sigma_j} \langle H_j, X\rangle \, d\mu_j + \int_{\partial\Sigma_j} \langle X, \nu_0\rangle ,$$

where ν_0 is the conormal vector field of $\partial\Sigma_j \subset \Sigma_j$.

Take $X(x) = \varphi^2\rho_{j+1}(\vec{x} - \vec{x}_0)$. Let $e_1, \cdots, e_j$ be an orthonormal basis for $T\Sigma_j$. Then

$$\begin{aligned} \mathrm{div}_{\Sigma_j}(X) &= \sum_{i=1}^{j} \langle \nabla_{e_i} X, e_i\rangle \\ &= j\varphi^2\rho_{j+1} + \left\langle \nabla_j(\varphi^2\rho_{j+1}), \vec{x} - \vec{x}_0 \right\rangle . \end{aligned}$$

Integrating in a ball $B_r(x_0) \subset \Omega$, we have

(3.46)

$$j\int_{\Sigma_j\cap B_r(x_0)} \varphi^2\rho_{j+1} d\mu_j \leq \int_{\Sigma_j\cap B_r(x_0)} \left(|\nabla_j(\varphi^2\rho_{j+1})| + |H_j|\varphi^2\rho_{j+1}\right) d\mu_j + r\int_{\partial(\Sigma_j\cap B_r(x_0))} \varphi^2\rho_{j+1} d\mu_{j-1}.$$

By the coarea formula,

$$\int_{\partial(\Sigma_j\cap B_r(x_0))} \varphi^2\rho_{j+1} d\mu_{j-1} \leq \frac{d}{dr}\int_{\Sigma_j\cap B_r(x_0)} \varphi^2\rho_{j+1} d\mu_j.$$

For a small number ε to be chosen later, take a finite covering $\mathcal{S} \subset \cup_{\alpha\in A} B_\alpha$, $B_\alpha = B_{r_\alpha}(x_\alpha)$, such that $\sum_{\alpha\subset A} r_\alpha^{n-1} < \varepsilon$. Take $V = \cup_{\alpha\in A} B_\alpha$.

Denote

$$L_\alpha(r) = \int_{B_r(x_\alpha)} \varphi^2\rho_{j+1} d\mu_j, \quad M_\alpha(r) = \int_{B_r(x_\alpha)} \left(|\nabla_j(\varphi^2\rho_{j+1})| + |H_j|\varphi^2\rho_{j+1}\right) d\mu_j.$$

Applying 3.46 to each $B_{r_\alpha}(x_\alpha)$, we get

$$jL_\alpha(r) \leq rM_\alpha(r) + r\frac{d}{dr}(L_\alpha(r)). \tag{3.47}$$

Let δ, ε_0 be small constants to be chosen later. Roughly speaking, δ is much larger than ε. All the small constants $\delta, \varepsilon, \varepsilon_0$ depend only on η and j.

Divide the index set A into two subsets A_1, A_2 defined by the following.

$$A_1 = \{\alpha \in A : \text{there exists } r'_\alpha \in [r_\alpha, \delta/5] \text{ such that } \varepsilon_0 L_\alpha(5r'_\alpha) \leq r'_\alpha M_\alpha(r'_\alpha)\},$$

$A_2 = A \setminus A_1$.

Also let $V_1 = \cup_{\alpha\in A_1} B_\alpha$, $V_2 = \cup_{\alpha\in A_2} B_\alpha$. We apply the 5-times covering lemma to V_1, V_2.

Lemma 3.20. *There exists $A'_1 \subset A_1$ with $\{B'_\alpha = B_{r'_\alpha}(x_\alpha) : \alpha \in A'_1\}$ pairwise disjoint, and*

$$\bigcup_{\alpha\in A_1} B'_\alpha \subset \bigcup_{\alpha\in A'_1} 5B'_\alpha.$$

For $\alpha \in A_1$, we have

$$\varepsilon_0 L_\alpha(5r'_\alpha) \leq r'_\alpha M_\alpha(r'_\alpha) \leq \frac{\delta}{5} M_\alpha(r'_\alpha).$$

Summing over the indices in A'_1, we get

$$\sum_{\alpha\in A'_1} \varepsilon_0 L_\alpha(5r'_\alpha) \leq \frac{\delta}{5} M_\alpha(r'_\alpha).$$

Using the fact that $V_1 \subset 5B'_\alpha$ and that B'_α are disjoint, we conclude

$$\varepsilon_0 \int_{V_1} \varphi^2\rho_{j+1} d\mu_j \leq \delta\int_{\Sigma_j\cap\Omega} \left(|\nabla_j(\varphi^2\rho_{j+1})| + |H_j|\varphi^2\rho_{j+1}\right) d\mu_j. \tag{3.48}$$

Or

$$\int_{V_1} \varphi^2 \rho_{j+1} d\mu_j \le \delta \varepsilon_0^{-1} \int_{\Sigma_j \cap \Omega} \left(|\nabla_j (\varphi^2 \rho_{j+1})| + |H_j| \varphi^2 \rho_{j+1} \right) d\mu_j. \tag{3.49}$$

We next treat $\alpha \in A_2$. By definition, for every $r \in [r_\alpha, \delta/5]$, we have

$$jL_\alpha(r) \le \varepsilon_0 L_\alpha(5r) + r \frac{d}{dr}(L_\alpha(r)). \tag{3.50}$$

Denote $\sigma_k = 5^k r_\alpha$, $k = 0, 1, 2, \cdots$. Then there exists some integer p such that $\sigma_{p-1} \le \delta/5 < \sigma_p$. Define $\Lambda_k = L_\alpha(\sigma_k)$, $k = 0, \cdots, p$. Observe that for $r \in [\sigma_k, \sigma_{k+1}]$,

$$\Lambda_k \le L\alpha(r) \le L_\alpha(5r) \le \Lambda(k+2).$$

By 3.50 we therefore have

$$jL_\alpha(r) \le \varepsilon_0 \Lambda_{k+2} \Lambda_k^{-1} L_\alpha(r) + r \frac{d}{dr}(L_\alpha(r)).$$

Integrate for r from σ_k to σ_{k+1}, we obtain that

$$5^{j - \varepsilon_0 \Lambda_{k+2} \Lambda_k^{-1}} \le \Lambda_{k+1} \Lambda_k^{-1}.$$

Denote $R_k = \Lambda_{k+1} \Lambda_k^{-1}$. Then above tells us that

$$R_k \ge 5^{j - \varepsilon_0 R_{k+1} R_k}.$$

Choose $\varepsilon_0 = 5^{-2j+2}$. The above implies that whenever $R_k \le 5^{j-1}$, we would have that

$$5^{j-1} \ge R_k \ge 5^{j - \varepsilon_0 R_{k+1} R_k},$$

and thus, $\varepsilon_0 R_k R_{k+1} \ge 1$, or $R_k R_{k+1} \ge 5^{2j-2}$. As a conclusion, for each k we have either

$$R_k \ge 5^{j-1} \text{ or } R_k R_{k+1} \ge 5^{2j-2}.$$

As a consequence we have either

$$\Lambda_p \Lambda_0^{-1} = R_{p-1} \cdots R_0 \ge 5^{p(j-1)} \text{ or } \Lambda_{p-1} \Lambda_0^{-1} = R_{p-2} \cdots R_0 \ge 5^{(p-1)(j-1)}.$$

In any case, this implies that

$$\begin{aligned} L\alpha(r_\alpha) &= \Lambda_0 \\ &\le 5^j \frac{r_\alpha^{j-1}}{\delta^{j-1}} \max\{\Lambda_p, \Lambda_{p-1}\} \\ &\le 5^j \frac{r_\alpha^{j-1}}{\delta^{j-1}} \int_{\Sigma_j \cap \Omega} \varphi^2 \rho_{j+1} d\mu_j. \end{aligned} \tag{3.51}$$

Summing over $\alpha \in A_2$ and using the fact that $\sum_{\alpha \in A} r_\alpha^{n-1} < \varepsilon$, we obtain that

$$\int_{V_2} \varphi^2 \rho_{j+1} d\mu_j \le 5^{j-1} \varepsilon \delta^{1-j} \int_{\Sigma_j \cap \Omega} \varphi^2 \rho_{j+1} d\mu_j. \tag{3.52}$$

Finally we choose the constants in such a way that

$$\varepsilon_0 = 5^{-2j+2}, \quad \delta < \eta \varepsilon_0, \quad \varepsilon < \eta 5^{1-j} \delta^{j-1},$$

and combine 3.49 and 3.52 to conclude that

$$\int_V \varphi^2 \rho_{j+1} d\mu_j \le \eta \left(\int_{\Sigma_j \cap \Omega} (|\nabla_j (\varphi^2 \rho_{j+1})| + |H_j| \varphi^2 \rho_{j+1}) d\mu_j + \int_{\Sigma_j \cap \Omega} \varphi^2 \rho_{j+1} d\mu_j \right).$$

□

4. Homogeneous minimal slicings

In this section we develop a partial regularity theory of minimal slicings. Parallel to the regularity theory of area minimizing hypersurfaces, we use the Federer's dimension reduction technique. Assume we have a minimal k-slicing

$$\Sigma_k \subset \cdots \subset \Sigma_n \subset \mathbb{R}^N,$$

and a point $x \in \Sigma_k \cap \mathcal{S}_k$ in the singular set. Inductively assume that the Hausdorff dimension of the singular set $\mathcal{S}_{k+1}$ of Σ_{k+1} is at most $k-2$. We need to prove that $\dim(\mathcal{S}_k) \leq k-3$. Note that the inductive assumption implies that $\dim(\mathcal{S}_k) \leq k-2$. Like the usual regularity theory where the tangent cone plays an essential role, we rescale the minimal slicing at x and study a homogeneous minimal slicings. To do so we will need to prove that the rescaled slicings converges in suitable sense, and that the limit is scaling invariant. We develop a monotonicity formula to guarantee such scaling invariance of the limit.

For a small radius σ and $j \geq k$, rescale $B_\sigma(x)$ to the unit ball in $\mathbb{R}^N$, and denote

$$\Sigma_{j,\sigma} = \sigma^{-1}(\Sigma_j - x).$$

On each Σ_j there exists a positive eigenfunction u_j of the quadratic form Q_j. We rescale it to be a function on $\Sigma_{j,\sigma}$ inductively. Assume that $u_{j+1,\sigma}, \cdots, u_{n-1,\sigma}$ have been defined, and that we have the corresponding weight function $\rho_{j+1,\sigma} = u_{j+1,\sigma} \cdots u_{n-1,\sigma}$. Then define $u_{j,\sigma}$ by letting

$$u_{j,\sigma}(y) = a_j u_j(x + \sigma y),$$

where a_j is properly chosen such that

$$\int_{\Sigma_{j,\sigma} \cap B_1(0)} u_{j,\sigma}^2 \rho_{j+1,\sigma} d\mu_j = 1.$$

To extract a converging subsequence of the pair $(\Sigma_{j,\sigma}, u_{j,\sigma})$ we need bounds on several quantities.

Proposition 4.1. *There exists a constant Λ that depends only on the minimal slicing but not on σ, such that*

- *The first eigenvalue bound $\lambda_{j,\sigma} \leq \Lambda$.*
- *The weighted volume bound $\mathrm{Vol}_{\rho_{j+1,\sigma}}(\Sigma_{j,\sigma} \cap B_{\frac{1}{2}}(0)) \leq \Lambda$.*
- *The quantity P defined as in 3.35 has an integral bound*

$$\int_{\Sigma_{j,\sigma} \cap B_{\frac{1}{2}}(0)} (1 + |A_j|^2 + \sum_{p=j+1}^{n-1} |\nabla_{j,\sigma} \log u_{p,\sigma}|^2) u_{j,\sigma}^2 \rho_{j+1,\sigma} d\mu_j \leq \Lambda.$$

Proof. The first eigenvalue bound is straightforward. Under the rescaling, the eigenvalue changes by

$$\lambda_{j,\sigma} = \sigma^2 \lambda_j,$$

and therefore is bounded.

For the weighted volume bound and the integral bound, we prove a stronger estimate inductively. In fact, we prove that for some $\delta > 0$ independent of σ the corresponding estimates hold on $\Sigma_{j,\sigma} \cap B_{\frac{1}{2}+\delta}$. Assume by induction that the same statement holds for $\Sigma_{j+1,\sigma}$. From the integral bound and the weighted volume bound we have that

$$\int_{\Sigma_{j+1,\sigma} \cap B^N_{\frac{1}{2}+\delta}(0)} u_{j+1}^2 \rho_{j+2} d\mu_{j+1} \leq \Lambda, \quad \int_{\Sigma_{j+1,\sigma} \cap B^N_{\frac{1}{2}+\delta}(0)} \rho_{j+2} d\mu_{j+1} \leq \Lambda.$$

Then by the Hölder's inequality

$$\int_{\Sigma_{j+1,\sigma} \cap B^N_{\frac{1}{2}+\delta}(0)} \rho_{j+1} d\mu_{j+1} \leq \Lambda.$$

By the coarea formula,

$$\int_{\Sigma_{j+1,\sigma}\cap B^N_{\frac{1}{2}+\delta}(0)} \rho_{j+1} d\mu_{j+1} = \int_0^{\frac{1}{2}+\delta} \left(\int_{\Sigma_{j+1,\sigma}\cap\partial B^N_r(0)} \frac{\rho_{j+1}}{|\nabla r|} d\mu_j \right) dr$$

$$\geq \int_0^{\frac{1}{2}+\delta} \mathrm{Vol}_{\rho_{j+1}}(\Sigma_{j+1,\sigma}\cap \partial B^N_r(0)) dr.$$

Therefore there exists some $\delta' \in (0, \delta/2)$ such that

$$\mathrm{Vol}_{\rho_{j+1}}(\Sigma_{j+1} \cap \partial B^N_{\frac{1}{2}+\delta'}(0)) \leq 2\Lambda/\delta.$$

Since $\Sigma_j \subset \Sigma_{j+1}$ minimizes the weighted volume $V_{\rho_{j+1}}$, its weighted volume inside ball $B^N_{\frac{1}{2}+\delta'}(0)$ is less than or equal to the portion of the sphere $\partial B^N_{\frac{1}{2}+\delta'}(0)$ with the same boundary. Therefore

$$\mathrm{Vol}_{\rho_{j+1}}(\Sigma_j \cap B^N_{\frac{1}{2}+\delta'}(0)) \leq \frac{1}{2}\mathrm{Vol}_{\rho_{j+1}}(\Sigma_{j+1} \cap \partial B^N_{\frac{1}{2}+\delta'}(0)) \leq \Lambda/\delta,$$

the desired bound for possibly different constants Λ and δ.

The integral bound is a localization of the coercivity estimate Proposition 3.17. Let $Q_{j,\sigma}$ be the quadratic form for the rescaled slicing. By proposition 3.17, for any function φ compactly supported on the regular part of $\Sigma_{j,\sigma}$,

$$\|\varphi\|^2_{1,j} \leq c(Q_j(\varphi,\varphi) + \|\varphi\|^2_{0,j}).$$

Note that the rescaled surfaces $\Sigma_{j,\sigma}$ have uniformly bounded geometry hence the constant c in the above inequality can be chosen independently of σ. Take a cutoff function ζ which is identically 1 on $B^N_{\frac{1}{2}+\delta'}(0)$ and is 0 near $\partial B^N_1(0)$. Since $Q_{j,\sigma}$ is a symmetric quadratic form, we have that, for any function $v \in \mathcal{H}_{j,0}$,

$$Q_{j,\sigma}(\zeta v, \zeta v) = Q_{j,\sigma}(\zeta^2 v, v) + \int_{\Sigma_{j,\sigma}} v^2 |\nabla_{j,\sigma}\zeta|^2 \rho_{j+1,\sigma} d\mu_j.$$

Take $v = u_{j,\sigma}$ first Dirichlet eigenfunction in the above inequality. We have that

$$Q_{j,\sigma}(\zeta^2 u_{j,\sigma}, u_{j,\sigma}) = \lambda_{j,\sigma} \int_{\Sigma_{j,\sigma}} \zeta^2 u^2_{j,\sigma}\rho_{j+1} d\mu_j \leq \lambda_{j,\sigma} \leq c$$

by the scaling of $u_{j,\sigma}$. Also

$$\int_{\Sigma_{j,\sigma}} u^2_{j,\sigma}|\nabla_{j,\sigma}\zeta|^2 \rho_{j+1,\sigma} d\mu_j \leq 16 \int_{\Sigma_{j,\sigma}} u^2_{j,\sigma}\rho_{j+1} d\mu_j \leq 16.$$

We then have $Q_{j,\sigma}(u_{j,\sigma}, u_{j,\sigma}) \leq c$. By Proposition 3.17 we conclude that

$$\int_{\Sigma_{j,\sigma}\cap B_{\frac{1}{2}+\delta'}(0)} (1 + |A_j|^2 + \sum_{p=j+1}^{n-1} |\nabla_{j,\sigma} \log u_{p,\sigma}|^2) u^2_{j,\sigma}\rho_{j+1,\sigma} d\mu_j \leq Q_{j,\sigma}(\zeta u_{j,\sigma}, \zeta u_{j,\sigma}) \leq c.$$

□

With these bounds, we wish to extract a subsequence $\sigma_i \to 0$ such that the slicings $(\Sigma_{j,\sigma_i}, u_{j,\sigma_i})$ converges. Moreover, we want the limit slicing to be invariant under such rescaling. This brings the following definition.

Definition 4.2. A minimal k-slicing

$$\Sigma_k \subset \cdots \Sigma_n \subset \mathbb{R}^N$$

is called a homogeneous minimal k-slicing if for each $j \geq k$, Σ_j is a cone, and u_j is homogeneous of some degree. That is, $u_j(\lambda x) = \lambda^{d_j} u_j(x)$, for every x and $\lambda > 0$.

We will derive two important monotonicity formulas that guarantee the existence of a homogeneous minimal slicing after rescaling.

4.1. Frequency function and monotonicity for eigenfunctions. Let C^k be a cone in the unit ball of $\mathbb{R}^N$, $\mathcal{S} \subset C$ be a closed set of singular points with Hausdorff dimension less or equal than $k-3$. On C^k consider a quadratic form

$$Q(\varphi, \varphi) = \int_C \left(|\nabla \varphi|^2 - q(x)\varphi^2\right) \rho d\mu.$$

We assume that $q(x)$ is a potential function that is in the form

$$q = \bar{q} + \operatorname{div}(X),$$

such that

$$|\bar{q}| + |X|^2 \leq P.$$

We further assume that the density function ρ is homogeneous of degree p and the potential q is homogeneous of degree -2:

$$\rho(\lambda x) = \lambda^p \rho(x), \quad q(\lambda x) = \lambda^{-2} q(x).$$

Note that these assumptions are satisfied by the quadratic form Q on a homogeneous minimal slicing.

Let u be a critical point of Q with respect to $\int_C \varphi^2 \rho d\mu$, and $u > 0$ on the regular part of C. Define the quantities

$$Q_\sigma(u) = \int_{C \cap B_\sigma(0)} (|\nabla u|^2 - q u^2) \rho d\mu, \tag{4.1}$$

$$I_\sigma(u) = \int_{C \cap \partial B_\sigma(0)} u^2 \rho d\mu. \tag{4.2}$$

The frequency function $N(\sigma)$ is defined by

$$N(\sigma) = \frac{\sigma Q_\sigma(u)}{I_\sigma(u)}. \tag{4.3}$$

The importance of the frequency function is the following

THEOREM 4.3. *$N(\sigma)$ is a monotone increasing function of σ. In fact,*

$$N'(\sigma) = \frac{2\sigma}{I_\sigma(u)^2} \left[I_\sigma(u_r) I_\sigma(u) - \left(\int_{C \cap \partial B_\sigma} u_r u \right)^2 \right],$$

where u_r denotes the radial derivative of u. As σ approaches to 0 the limit $N(\sigma)$ exists and is finite. Moreover $N(\sigma)$ is constant if and only if u is homogeneous of degree $N(0)$.

PROOF. We first derive two formulas by taking variations with respect to two deformations. The advantage of this variational approach is that it works even in presence of singularities. Let $\zeta(r)$ be a nonnegative decreasing function supported in $[0, \sigma]$. The precise choice of ζ will be specified later. We describe the first deformation. Let $X = \zeta(|x|)x$ be a vector field in $\mathbb{R}^N$ where x is the position vector. The flow F_t of X then preserves the cone C, and hence the function $u_t = u \circ F_t$ is a valid function in the variational characterization for Q. Since u is a critical point, we have

$$0 = \left.\frac{d}{dt}\right|_{t=0} Q_\sigma(u_t) = \left.\frac{d}{dt}\right|_{t=0} \int_{C \cap B_\sigma(0)} \left(|\nabla_t u|^2 - (q \circ F_t) u^2\right) \rho \circ F_t d\mu_t.$$

Note that here we have used a change of variable. ∇_t and μ_t denotes the gradient operator and the volume measure with respect to the pull back metric $F_t^*(g)$ where g is the induced metric on C. Differentiating each term with respect to t and evaluating each derivative at $t=0$, we have

$$\begin{aligned}\frac{d}{dt}|\nabla_t u|^2 &= -\langle \mathcal{L}_X g, du\otimes du\rangle\\ &= -\left\langle 2r\zeta'(r)(dr\otimes dr)+2\zeta g, du\otimes du\right\rangle\\ &= -2r\zeta'(r)(u_r)^2-2\zeta|\nabla u|^2.\end{aligned}$$

For a homogeneous function f of degree d, $\frac{d}{dt}(f\circ F_t)=X(f)=d\zeta f$. We therefore have that

$$\frac{d}{dt}(q\circ F_t)=-2\zeta q,\quad \frac{d}{dt}(\rho\circ F_t)=p\zeta\rho.$$

For the volume form we have

$$\left.\frac{d}{dt}\right|_{t=0} d\mu_t=\operatorname{div}_C(X)d\mu=(r\zeta'(r)+k\zeta)d\mu.$$

Collecting all the terms we conclude that

$$0=\int_{C\cap B_\sigma(0)}[(p+k-2)\zeta(|\nabla u|^2-qu^2)+r\zeta'(|\nabla u|^2-2u_r^2-qu^2)]\rho d\mu.$$

We now choose ζ to be a function which is 1 in $C\cap B_{\sigma-\varepsilon}(0)$ and is 0 outside $C\cap B_\sigma(0)$, and let $\varepsilon\to 0$. Then the above implies that

$$\begin{aligned}(p+k-2)Q_\sigma(u)&=\sigma\int_{C\cap\partial B_\sigma(0)}(|\nabla u|^2-2u_r^2-qu^2)\rho d\mu_{k-1}\\ &=\sigma\frac{dQ_\sigma(u)}{d\sigma}-2\sigma\int_{C\cap\partial B_\sigma(0)}u_r\rho d\mu_{k-1}.\end{aligned}\tag{4.4}$$

Now we describe the second deformation. Let $u_t=(1+t\zeta(r))u$. Then

$$0=\left.\frac{d}{dt}\right|_{t=0}Q_\sigma(u_t)=2\int_{C\cap B_\sigma(0)}\left(\langle\nabla u,\nabla(\zeta u)\rangle-q\zeta u^2\right)\rho d\mu.$$

Like before let ζ approach the characteristic function of $B_\sigma(0)$ we conclude

$$Q_\sigma(u)=\int_{C\cap\partial B_\sigma(0)}uu_r\rho d\mu_{k-1}.\tag{4.5}$$

Next we directly calculate the derivative $\frac{d}{d\sigma}I_\sigma(u)$. Note that $u\in\mathcal{H}_k(C\cap B_1(0))$, and that the cone structure guarantees that on each sphere $C\cap\partial B_\sigma(0)$ the singular set is of codimension two. Therefore the function $I_\sigma(u)$ is a C^1 function of σ. Taking derivative directly, we have

$$\sigma\frac{d}{d\sigma}I_\sigma(u)=2\sigma\int_{C\cap\partial B_\sigma}uu_r\rho d\mu_{k-1}+(p+k-1)\int_{C\cap\partial B_\sigma}u^2\rho d\mu_{k-1}.\tag{4.6}$$

We are ready to prove the theorem by combining 4.4, 4.5 and 4.6. First we have

$$N'(\sigma)=I_\sigma(u)^{-2}[(Q_\sigma+\sigma Q'_\sigma)I_\sigma-\sigma Q_\sigma I'_\sigma].$$

Substituting in the expression involving derivatives,

$$\begin{aligned}N'(\sigma)=&I_\sigma^{-2}[(Q_\sigma+(p+k-2)Q_\sigma)I_\sigma-Q_\sigma(p+k-1)I_\sigma]\\ &+2\sigma I_\sigma^{-2}\left(\int_{C\cap\partial B_\sigma(0)}u_r^2\rho d\mu_{k-1}-Q_\sigma^2 I_\sigma\right).\end{aligned}$$

The first term on the right is 0, and we use 4.5 in the second term and conclude

$$N'(\sigma) = 2I_\sigma(u)^{-1}\left(I_\sigma(u)I_\sigma(u_r) - \int_{C\cap\partial B_\sigma(0)} u_r u d\mu_{k-1}\right),$$

as desired.

To prove that $\lim_{\sigma\to 0} > -\infty$ we look at the expression

$$\bar{I}_\sigma(u) = \frac{\int_{C\cap\partial B_\sigma(0)} u^2\rho d\mu_{k-1}}{\int_{C\cap\partial B_\sigma(0)} \rho d\mu_{k-1}}.$$

Set $t = \log\sigma$. By direct calculation we have

$$N(\sigma) = \frac{1}{2}\frac{d}{dt}\log \bar{I}_\sigma(u).$$

Therefore $\log \bar{I}_\sigma(u)$ is a convex function of $t = \log\sigma$.

On the other hand, since $\int_{C\cap B_1(0)} u^2\rho d\mu_k < c$, by the coarea formula, for each $\sigma_1 \in [0, \frac{1}{2}]$ there exists some $\sigma \in [\sigma_1, 2\sigma_1]$ such that $\int_{C\cap\partial B_\sigma(0)} u^2\rho d\mu_{k-1} < c/\sigma$. We then have that

$$\bar{I}_\sigma(u) < \frac{c\sigma^{-1}}{\sigma^{p+k-1}\int_{C\cap\partial B_1(0)}\rho(\xi)d\mu_{k-1}(\xi)} < c\sigma^{-K},$$

for some K large enough. Hence there exists a sequence of σ_i converging to 0 such that

$$\bar{I}_{\sigma_i}(u) < c\sigma_i^{-K},$$

or

$$\log \bar{I}_{\sigma_i}(u) < -ct_i.$$

The function $\log \bar{I}_\sigma(u)$ is then a convex function of t which lies below a linear function $-ct$, hence its derivative is bounded from below by some negative constant $-c_0$. That is, $N(\sigma) \geq -c_0$ for all $\sigma > 0$.

Finally if $N(\sigma) = N(0)$ is a constant then we have equality in the Schwartz inequality for each σ. Denote $\xi = \frac{x}{|x|}$ and view the function u as a function of $(|x|, \xi)$. Equality in the Schwartz inequality then implies that

$$u_r(\sigma, \xi) = f(\sigma)u(\sigma, \xi)$$

for some function $f(\sigma)$. This implies that

$$\sigma f(\sigma) = \frac{\sigma\int_{C\cap\partial B_\sigma(0)} u u_r \rho d\mu_{k-1}}{\int_{C\cap B_\sigma(0)} u^2\rho d\mu_{k-1}} = N(0).$$

It then follows that $u_r = r^{-1}N(0)u$. Therefore u is a homogeneous function of degree $N(0)$.

□

4.2. Monotonicity formula for weighted minimal surfaces. Consider the first slice Σ_k where a point p is singular. In other words, $p \in \mathcal{S}_k$ but $p \in \mathcal{R}_{k+1}$. By the monotonicity formula for the first eigenfunction we know the rescaled surfaces $\Sigma_{k,\sigma}$ converges to a minimal cone with the weight function ρ_{k+1} converging to a homogeneous function. To study the regularity of the limit surface around the point p we need to extend the usual monotonicity formula for minimal surfaces to surfaces minimizing weighted volume. Let C be a $k+1$ dimensional cone in $\mathbb{R}^N$ with a singular set $\mathcal{S}$ of Hausdorff dimension at most $k-2$. Let ρ be a positive homogeneous function on C of degree p. Assume ρ is positive and smooth on the regular set of C.

THEOREM 4.4. *Assume $\Sigma \subset C$ is a hypersurface that minimizes the weighted volume V_ρ. Then we have the monotonicity formula*

$$\frac{d}{d\sigma}\left(\sigma^{-k-p}\operatorname{Vol}_\rho(\Sigma\cap B_\sigma(0))\right)=\int_{\Sigma\cap\partial B_\sigma(0)} r^{-p-k-2}|x^\perp|^2 d\mu_{k-1},$$

where $x^\perp$ is the component of the position vector x othogonal to Σ. In particular, the function $\sigma^{-k-p}\operatorname{Vol}_\rho(\Sigma\cap B_\sigma(0))$ is increasing, and is constant only if Σ is a cone.

PROOF. For a vector field X in $\mathbb{R}^N$, let F_t be the one parameter diffeomorphism generated by X. The first variation formula for the weighted volume reads

$$\left.\frac{d}{dt}\right|_{t=0}\int_\Sigma (\rho\circ F_t)d\mu_k=0,$$

where μ_t is the volume with respect to the pull back metric $F_t^*(g)$, and g is the induced metric on Σ in $\mathbb{R}^N$. Differentiating and evaluating at $t=0$ implies that

$$\int_\Sigma (X(\rho)+\operatorname{div}_\Sigma(X))d\mu_k=0.$$

Choose a function $\zeta(r)$which is decreasing, nonnegative, and equal to 0 for $r>\sigma$. Define $X=\zeta(|x|)x$, where x is the position vector field. Since ρ is homogeneous of degree p we have that $X(\rho)=p\zeta\rho$. And $\operatorname{div}_\Sigma(X)=k\zeta+r^{-1}\zeta'|x^T|^2$, where x^T denotes the component of x tangential to Σ. Therefore

$$\int_\Sigma [(p+k)\zeta+r^{-1}\zeta'|x^T|^2]\rho d\mu_k=0.$$

For a small $\varepsilon>0$, take ζ to be 1 in $B_{\sigma-\varepsilon}(0)$ and 0 outside B_σ into the above equality. Letting $\varepsilon\to 0$, we have

$$\begin{aligned}(p+k)\operatorname{Vol}_\rho(\Sigma\cap B_\sigma(0))&=\int_{\Sigma\cap\partial B_\sigma(0)} r^{-1}|x^T|^2\rho d\mu_{k-1}\\ &=\sigma\frac{d}{d\sigma}\int_{\Sigma\cap\partial B_\sigma(0)}\rho d\mu_{k-1}-\int_{\Sigma\cap\partial B_\sigma(0)} r^{-1}|x^\perp|^2\rho d\mu_{k-1}\\ &=\sigma\frac{d}{d\sigma}\operatorname{Vol}_\rho(\Sigma\cap B_\sigma(0))-\int_{\Sigma\cap\partial B_\sigma(0)} r^{-1}|x^\perp|^2\rho d\mu_{k-1}.\end{aligned}$$

Note that we have used the fact that $|x|^2=|x^T|^2+|x^\perp|^2$, and that x is tangential to C since C is a cone. The monotonicity formula is then obtained by rearranging terms properly. □

5. Top dimensional singularities

Given a minimal slicing, let p be a point in the singular set. Assume that Σ_m is the first singular slice at p. In other words, $p\in\mathcal{S}_m$ and $p\in\mathcal{R}_{m+1}$. From the previous section we know that after rescaling at p a limit homogeneous minimal cone exists. Since Σ_j is regular at p for $j>m$, the homogeneous minimal slicing is given by

$$C^m\subset\mathbb{R}^{m+1}\subset\cdots\subset\mathbb{R}^n,$$

where C^m is a volume minimizing cone. Note that for this minimal slicing all the weight functions $u_j=1$, for $j>m$, and that the density function $\rho_{m+1}=1$. We first need a technical proposition that will be used several times.

THEOREM 5.1. *Assume $C^m\subset\mathbb{R}^{m+1}$ is a volume minimizing cone which is not a hyperplane, u_m is a positive minimizer for the quadratic form Q_m:*

$$Q_m(\varphi,\varphi)=\int_{C\cap B_1(0)} S_m(\varphi,\varphi)+\frac{3}{8}\int_{C\cap B_1(0)}|A_C|^2\varphi^2 d\mu_m,$$

where S_m is the stability operator. Assume u_m is homogeneous of degree d. Then there exists a constant $c = c(m) > 0$ that only depends on the dimension such that $d < -c$.

The proof of the theorem is divided into several steps. First we prove the following lemma.

LEMMA 5.2. *Under the same assumption as in Theorem 5.1, the homogeneity degree d is negative.*

PROOF. Suppose $u_m = r^d v(\xi)$, v is a function defined on the cross section $\Sigma = C \cap S^m(1)$. Note that since $\rho_{m+1} = 1$ the function u_m is in the usual Sobolev space $W^{1,2}(C)$, and v is in $W^{1,2}(\Sigma)$. The fact that C is an area minizing hypercone implies that the singular set of C is of Hausdorff dimension at most $m-7$, and that the singular set of $\Sigma \subset S^m$ is of dimension at most $m-8$. Since u_m is a minimizer of the functional

$$\begin{aligned} Q(\varphi, \varphi) &= \int_{C \cap B_1(0)} S_m(\varphi, \varphi) + \frac{3}{8} \int_{C \cap B_1(0)} |A_C|^2 \varphi^2 d\mu_m \\ &= \int_{C \cap B_1(0)} \left(|\nabla \varphi|^2 - \frac{5}{8} |A_C|^2 \varphi^2 \right) d\mu_m, \end{aligned}$$

u weakly solves the equation

$$\Delta_C u_m + \frac{5}{8} |A_C|^2 u_m = 0.$$

By separation of variables v weakly solves the equation on Σ:

$$\Delta_\Sigma v + d(d+m-1)v + \frac{5}{8} |A_C|^2(\xi) v = 0.$$

In other words, for any $\psi \in W^{1,2}(\Sigma)$,

$$\int_\Sigma \left(\nabla v \cdot \nabla \psi - \frac{5}{8} |A_C|^2 v \psi \right) d\mu_{k-1} = d(d+m-1) \int_\Sigma v \psi d\mu_{k-1}.$$

Let $\mu = d(d+m-1)$. Since the singular set of $\Sigma \subset S^m(1)$ is of dimension at most $m-8$, the constant function 1 is in $W^{1,2}(\Sigma)$. Substitute $\psi = 1$ in the equation above, we have that

$$\mu \int_\Sigma v d\mu_{k-1} = -\frac{5}{8} \int_\Sigma |A_C|^2 v < 0,$$

since $v > 0$ and C is not a hyperplane. This proves $\mu = d(d+m-1) < 0$. Therefore $d < 0$. □

To prove that d can be uniformly bounded from above by some negative constant, we use a compactness argument.

LEMMA 5.3. *The space*

$$\mathcal{M} = \{\textit{Area minimizing cones } C^m \subset \mathbb{R}^{m+1} \textit{ which is not a hyperplane}\}$$

is compact in flat norm.

PROOF. To see this, we first note that the volume density $\mathrm{Vol}(\Sigma \cap B_1(0))$ is uniformly bounded by comparision with the unit sphere. Hence by the compactness theorem of volume minimizing currents any sequence C_i of minimizing cones must have a converging subsequence. Now if each C_i is not a hyperplane and $C_i \to C$ in flat norm, then C is not a hyperplane. Otherwise for any $\varepsilon_0 > 0$ and i large $|\mathrm{Vol}(C_i \cap B_1(0)) - \omega_m| < \varepsilon_0$, hence by the Allard theorem C_i must be regular, that is, C_i must be a hyperplane, contradiction. □

The next step is to prove that the quadratic form Q_i converges as the minimizing cones C_i converge to C. In fact, we prove a more general result that will be used for later purposes.

Proposition 5.4. *Assume $\{\Sigma_i\}$ is a sequence of area minimizing hypersurfaces of a Riemannian manifold M such that Σ_i converges to Σ. Let Q_i, Q denote the quadratic forms defined on Σ_i, Σ, respectively, and u_i be the first eigenfunctions of Q_i. Let U be an open subset of M. Then*

- *Σ_i converges to in C^2 norm to Σ locally in $\bar{U}$ on the complement of the singular set of Σ*
- $\lim_{i\to\infty} \mathrm{Vol}(\Sigma_i \cap U_i) = \mathrm{Vol}(\Sigma \cap U)$
- $\lim_{i\to\infty} \|u_i\|_{W^{1,2}(U_i)} = \|u\|_{W^{1,2}(U)}$
- *For any smooth compactly supported function φ on M,*

$$\lim_{i\to\infty} Q_i(\varphi u_i, \varphi u_i) = Q(\varphi u, \varphi u).$$

where U_i is a sequence of compact subdomains of U with $U_i \subset U_{i+1} \subset U$ and $\cup U_i = U$.

Note that this combined with the previous two lemmas conclude the proof of Theorem 5.1: If u_i on C_i is the minimizer of Q_i, normalized $\|u_i\|_{L^2(C_i\cap B_1(0))} = 1$, then by the fourth item in the above proposition the limit function u is the minimizer of Q on C. In particular, the homogeneity degree is continuous under the flat norm convergence. Since it is negative for every non-planar area minimizing cone, it is uniformly bounded from above by some negative number depending only on m.

To prove this proposition, we will implement an important idea that will be used to handle the general compactness theorem. The difficult part is the fourth statement, the convergence of the quadratic form in presence of singular set. One needs to prove that the eigenfunction u_i do not concentrate on the singular set. Recall that by Proposition 3.19 we do have an weighted L^2 nonconcentration result for functions in the weighted Sobolev space. The remaining question is then to control the Dirichlet integral $\|\nabla u_i\|_{L^2}$. The proof is carried out with the help of cut off functions that isolate the singular sets, as will be illustrated below.

Proof. The first and second statements follow from standard theory of area minimizing surfaces. We focus on the proof of the convergence of $W^{1,2}$ norm and the quadratic forms. First we prove the L^2 convergence. To do so, observe that the singular set $\mathcal{S}_i$ of Σ_i convergence to $\mathcal{S}$ in the sense that for any $\varepsilon > 0$, $\mathcal{S}_i$ is contained in an ε neighborhood of $\mathcal{S}$ as i approaches to infinity, by the Allard theorem. Therefore u_i converges uniformly to u on compact subsets of $\Sigma_i \setminus \mathcal{S}_i$, where we write Σ_i locally as a normal graph over Σ and compare corresponding values of u_i to u. In particular, if W is a compact subdomain of $U \cap \mathcal{R}_i$ we have convergence of L^2 norms of u_i to the corresponding L^2 norm of u. Now apply Proposition 3.19 with $\mathcal{S} = \mathcal{S}(C)$, where the Hausdorff dimension of $\mathcal{S}$ is at most $m-7$. We may find an open neighborhood V of $\mathcal{S} \cap \bar{U}$ such that for a fixed small number η and sufficiently large i, $\mathcal{S}_i \cap \bar{U} \subset V$, and

$$\begin{aligned}\int_{\Sigma_i\cap V} u_i^2 d\mu_m &\le \eta \int_{\Sigma_i\cap U} [|\nabla u_i|^2 + (1+P_i)u_i^2] d\mu_m \\ &\le \eta C_0,\end{aligned}$$

here C_0 is an upper bound of the $W^{1,2}$ norms of u_i.

Choosing η arbitrarily small, we have a uniform control of the L^2 norm of u_i over a small open neighborhood of $\mathcal{S}$. Combine this with the L^2 convergence of u_i to u on compact subsets of $U - \mathcal{S}$ we have the L^2 convergence, namely

$$\|u_i\|_{L^2(U_i)} \to \|u\|_{L^2(U)}.$$

We next deal with the Dirichlet integral. Recall that the singular set $\mathcal{S}$ of Σ is of codimension at least 7. We first construct a Lipschitz function ψ that isolates the singular set $\mathcal{S}$. For any given constants $\varepsilon, \delta > 0$ and $a \in (0,7)$, cover $\mathcal{S}$ by finitely balls $\{B_{r_i}(x_i)\}_{i=1}^K$ with $\sum r_i^{m-7} < \varepsilon$ and

$\varepsilon \geq r_1 \geq r_2 \geq \cdots r_K$ such that

$$\sum_{j=1}^{M} r_j^{m-7} < \delta.$$

Let

$$\beta_j(x) = \begin{cases} 1, & \text{if } x \in B_j \\ 2 - r_j^{-1}|x - x_j|, & \text{if } x \in 2B_j - B_j \\ 0 & \text{if } x \in M \setminus 2B_j. \end{cases}$$

And denote $\psi_1(x) = \beta_1(x)$,

$$\psi_j(x) = \max\{\beta_j - \max\{\beta_1, \beta_2, \cdots, \beta_{j-1}\}, 0\}, \quad 2 \leq j \leq K.$$

Then ψ_j is supported in $2B_j$ and define

$$\psi(x) = \sum_{j=1}^{K} \psi_j(x) = \max_j \psi_j(x).$$

We then have that $\psi = 1$ in a neighborhood of $\mathcal{S}$, $\psi = 0$ for points that are of 2ε away from $\mathcal{S}$, and

$$\int_\Sigma |\nabla\psi|^a d\mu_m < \sum_{j=1}^{M} r_j^{m-a} < \delta\varepsilon^{7-a}.$$

We prove this implies that

$$\int_\Sigma |\nabla\psi|^2 u^2 d\mu_m \leq c\varepsilon^{5/16}. \tag{5.1}$$

To do so, observe that u satisfies an equation in the form

$$\Delta u + \frac{5}{8}|A|^2 u + qu = 0,$$

where q is a bounded function. On the other hand stability inequality implies that

$$\int_\Sigma |A|^2 \varphi^2 d\mu_m \leq \int_\Sigma (|\nabla\varphi|^2 + c\varphi^2) d\mu_m.$$

Replace φ by $u^{8/5}\varphi$ and use the equation to obtain

$$\int_\Sigma |\nabla(u^{8/5}\varphi)|^2 d\mu_m \leq c \int_\Sigma u^{16/5}(|\nabla\varphi|^2 + \varphi^2) d\mu_m.$$

By the Michael-Simon Sobolev inequality, we obtain

$$\left(\int_\Sigma u^{16m/5(m-2)} \varphi^{m/(m-2)} d\mu_m\right)^{m-2/m} \leq c \int_\Sigma u^{16/5}(|\nabla\varphi|^2 + \varphi^2) d\mu_m.$$

Since $\dim(\mathcal{S}) \leq m - 7$ we may choose φ properly approximating the constant function 1, and combine with the fact that $\|u\|_{L^2(\Sigma)} \leq c$ to conclude

$$\int_\Sigma u^{\frac{16m}{5(m-2)}} d\mu_m \leq c.$$

We then apply the Hölder inequality to obtain

$$\int_\Sigma |\nabla\psi|^2 u^2 d\mu_m \leq \|\nabla\psi\|^2_{\frac{16m}{3m+10)}} \|u\|^2_{\frac{16m}{5(m-2)}}.$$

Setting $a = \frac{16m}{3m+10} < 7$, we have that

$$\int_\Sigma |\nabla\psi|^2 u^2 d\mu_m \leq c\varepsilon^{5/16}.$$

Now we are ready to prove the convergence of the Dirichlet integral. On each Σ_i choose a function ψ_i satisfying 5.1. We prove that for any smooth function φ compactly supported in U,

$$\lim_{i\to\infty} Q_i(\varphi u_i, \varphi u_i) = Q(\varphi u, \varphi u).$$

Denote $\varphi_i = \psi_i\varphi$.

We have

$$\begin{aligned} Q_i(\varphi u_i, \varphi u_i) &= Q_i(\varphi_i u_i, \varphi_i u_i) \\ &\quad + 2Q_i(\varphi_i u_i, (1-\psi_i)\varphi u_i) + Q_i((1-\psi_i)\varphi u_i, (1-\psi_i)\varphi u_i). \end{aligned}$$

The last two terms involve the function $1-\psi_i$, hence is compactly supported in the regular set $\mathcal{R}_i$. Thus the last two terms converges to the corresponding terms of the limit Q. It remains to prove that

$$\lim_{i\to\infty} Q_i(\varphi_i u_i, \varphi_i u_i) = Q(\psi\varphi u, \psi\varphi u).$$

Since u_i is an eigenfunction of the quadratic form Q_i, by the general variational principle we have

$$\begin{aligned} Q_i(\varphi_i u_i, \varphi_i u_i) &= Q_i(\varphi_i^2 u_i, u_i) + \int_{\Sigma_i} |\nabla\varphi_i|^2 u_i^2 d\mu_m \\ &= \lambda_i \int_{\Sigma_i} \varphi_i^2 u_i^2 d\mu_m + \int_{\Sigma_i} |\nabla\varphi_i|^2 u_i^2 d\mu_m \end{aligned}$$

The first term on the right side converges to zero as $\varepsilon \to 0$. The first eigenvalues λ_i is bounded. The integrand φ_i is abounded function supported in an ε neighborhood of the singular set, and by the L^2 nonconcentration result the integral converges to 0 as $\varepsilon \to 0$. Now the second term can further split into

$$\int_{\Sigma_i} |\nabla\varphi_i|^2 u_i^2 d\mu_m \le \int_{\Sigma_i} |\nabla(\psi_i)|^2 \varphi^2 u_i^2 d\mu_m + \int_{\Sigma_i} \psi_i^2 |\nabla\varphi|^2 u_i^2 d\mu_m.$$

We deal with these two terms separately. The first term converges to 0 because of 5.1. The second term converges to 0 because of the fact that ψ_i is supported in an ε neighborhood of $\mathcal{S}_i$ and the L^2 nonconcentration result.

The convergence is then established by letting $\eta \to 0$ and $\varepsilon \to 0$.

□

6. Compactness of minimal slicings

We now establish the general compactness result of minimal slicings. We have seen from the previous section that an essential issue of this is to prevent the concentration of the eigenfunctions u_i of Q_i on the singular set. The argument we use here is a generalization of the one we used to establish the convergence of Q_i for area minimizing hypersurfaces. The main technical tool we use is the existence of a proper function ψ_k on Σ_k that isolates the singular set $\mathcal{S}_k$. Let us elaborate this point.

Assume $\Sigma_k \subset \cdots \subset \Sigma_n \subset \mathbb{R}^N$ is a minimal k-slicing. Inductively we assume that $\dim(\mathcal{S}_k) \le k-2$. Take an open subset $U \subset \mathbb{R}^N$.

PROPOSITION 6.1. *There exists a function $\Psi_k \ge 1$ locally Lipschitz on $\mathcal{R}_k \cap U$, proper on $\mathcal{R} \cap \bar{U}$ and*

$$\int_{\Sigma_k} u_k^2 |\nabla\psi_k|^2 \rho_{k+1} d\mu_k < \Lambda,$$

where Λ is the upper bound appeared in Proposition 4.1.

PROOF. For $x \in \mathcal{R}_k$ define

$$\Psi_k(x) = \max\{1, \log u_k(x), \cdots, \log u_{n-1}(x)\}.$$

Then

$$u_k^2 |\nabla_k \Psi_k|^2 \leq u_k^2 \left(\sum_{p=k}^{n-1} |\nabla_k \log u_p|^2 \right).$$

Consequently

$$\begin{aligned} &\int_{\Sigma_k \cap U} u_k^2 |\nabla_k \Psi_k|^2 \rho_{k+1} d\mu_k \\ &\quad \leq \int_{\Sigma_k \cap U} u_k^2 \left(\sum_{p=k}^{n-1} |\nabla_k \log u_p|^2 \right) \rho_{k+1} d\mu_k \\ &\quad \leq \Lambda. \end{aligned}$$

The difficulty is to prove the properness of the function Ψ_k, that is, at any sequence of points x_i converging to $x_0 \in \mathcal{S}_k$, $\Psi_k(x_i) \to \infty$. For $x_0 \in \mathcal{S}_k$ there exists an integer m such that Σ_j is regular at x_0 for $j \geq m+1$, and Σ_m is singular at x_0. We prove $u_m(x_i) \to \infty$. It can be implied by the following

LEMMA 6.2. *There exists $\alpha \in (0,1)$ that only depends on the minimal slicing, such that if $\sigma > 0$ is a small radius then*

$$\inf_{B_{\alpha\sigma}(x_0) \cap \Sigma_{m+1}} u_m > 2 \inf_{B_\sigma(x_0) \cap \Sigma_{m+1}} u_m.$$

To prove the lemma, we first need a result by Bombieri-Giusti:

THEOREM 6.3 ([**BG72**]). *Assume T^m is an area minimizing surface in $\mathbb{R}^{m+1}$, and a function u satisfies $\Delta u \leq 0$, $u > 0$ almost everywhere. Then there exists a constant $c > 0$ that only depends on the dimension, such that for every point $y \in T$*

$$u(y) \geq \frac{c}{r^m} \int_{T \cap B_r(y)} u d\mu_m.$$

In particular, if u is a function on an area minimizing cone $C^m \subset \mathbb{R}^{m+1}$ that satisfies

$$\Delta u + \frac{5}{8} |A_C|^2 u = 0, \quad u > 0,$$

then $\inf_{B_\sigma(x_0) \cap C_m} > 0$.

The proof of lemma relies on a blow-up argument. Let us suppose the contrary, that there exists a sequence $\sigma_i \to 0$ and $\alpha = \frac{\alpha_0}{2}$ such that

$$\inf_{B_{\alpha\sigma_i}(x_0) \cap \Sigma_{m+1}} u_m < 2 \inf_{B_{\sigma_i}(x_0) \cap \Sigma_{m+1}} u_m,$$

where α_0 is chosen later.

Rescale σ_i to 1 and consider

$$\Sigma_{m,\sigma_i} = \sigma_i^{-1}(\Sigma_m - x_0).$$

By the monotonicity formula and the compactness result for area minimizing cones, Σ_{m,σ_i} converges to a cone $C_m \subset \mathbb{R}^{m+1}$ along with u_{m,σ_i} converging to a homoegenous u_m^C minimizing Q_C. Moreover, the assumption implies that

$$\inf_{B_\alpha(0)} u_m^C \leq 2 \inf_{B_1(0)} u_m^C.$$

On the other hand, the mean value inequality implies that $\inf_{B_r}(0)u_m^C > 0$. By Theorem 5.1 the function u_m^C is homogeneous of degree less than $-c(m)$, hence there exists a real number α_0 such that

$$\inf_{B_{\alpha_0}(0)\cap C_m} u_m^C > 2 \inf_{B_1(0)\cap C_m} u_m^C,$$

contradiction. □

We can state the compactness for minimal slicings satisfying the assumptions in Proposition 4.1. Assume that $\Sigma_k^{(i)}$ converge to Σ_k in C^k topology on compact subsets of regular sets. By induction, the singular set $\mathcal{S}_k$ of Σ_k has Hausdorff dimension less than or equal to $k-2$.

The proper functions Ψ_k can be used to study the convergence in two aspects.

PROPOSITION 6.4. *Let φ be a smooth function of compact support in U. Then*

$$\lim_{i\to\infty} Q_k^{(i)}(\varphi u_k^{(i)}, \varphi u_k^{(i)}) = Q_k(\varphi u_k, \varphi u_k).$$

In other words, the quadratic forms $Q_k^{(i)}$ converges to Q_k.

PROOF. Observe first that the L^2 convergence is given by the nonconcentration Proposition 3.19. The rest of the proof here closely resembles that of Theorem 5.1. For R large, we take a cut off function $\gamma(t)$ which is 0 for $t < R$, is 1 for $t > 2R$, and $|\gamma'| < \frac{2}{R}$. Then the function $\psi_k^{(i)} = \gamma(\Psi_k^{(i)})$ plays the same role as the function ψ_i in the proof of Proposition 5.4. Write $\varphi_i = \gamma(\Psi_k^{(i)})\varphi$, and split

$$\begin{aligned} Q_k^{(i)}(\varphi u_k^{(i)}, \varphi u_k^{(i)}) &= Q_k^{(i)}(\varphi_i u_k^{(i)}, \varphi_i u_k^{(i)}) \\ &\quad + 2Q_k^{(i)}(\varphi_i u_k^{(i)}, (1-\gamma(\Psi_k^{(i)}))\varphi u_k^{(i)}) + Q_k^{(i)}((1-\gamma(\Psi_k^{(i)}))\varphi u_k^{(i)}, (1-\gamma(\Psi_k^{(i)}))\varphi u_k^{(i)}). \end{aligned}$$

The last two terms involves functions that are compactly supported on the regular set, hence converges to the corresponding terms of Q_k. The first term can be dealt with similar by

$$Q_k^{(i)}(\varphi_i u_k^{(i)}, \varphi_i u_k^{(i)}) = \lambda_1^{(i)} \int_{\Sigma_k^{(i)}} \varphi_i^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k + \int_{\Sigma_k^{(i)}} |\nabla \varphi_i|^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k.$$

The first term on the right hand side converges to 0, since λ_1 is uniformly bounded and the weighted L^2 norm does not concentrate near the singular set. The second term on the right hand side can be further split into two parts

$$\int_{\Sigma_k^{(i)}} |\nabla(\gamma(\Psi_k^{(i)}))|^2 \varphi^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k + \int_{\Sigma_k^{(i)}} \gamma(\Psi_k^{(i)})^2 |\nabla \varphi|^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k.$$

The second term in the above expression can be estimated by the L^2 nonconcentration and the fact that $\psi_k^{(i)} = \gamma(\Psi_k^{(i)})$ is supported in a small neighborhood of $\mathcal{S}_i$. For the first term, we have that

$$\begin{aligned} \int_{\Sigma_k^{(i)}} |\nabla(\gamma(\Psi_k^{(i)}))|^2 \varphi^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k &= \int_{\Sigma_k^{(i)}} |\gamma'|^2 |\nabla \Psi_k^{(i)}|^2 \varphi^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k \\ &\le cR^{-2} \int_{\Sigma_k^{(i)}} |\Psi_k^{(i)}|^2 (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k \end{aligned}$$

By the construction of $\Psi_k^{(i)}$ this term converges to 0 as $R \to \infty$. Therefore the convergence of $Q_k^{(i)}$ is established as we choose $\varepsilon \to 0$ and $R \to \infty$. □

PROPOSITION 6.5. *Assume that $x_0 \in \mathcal{S}_{k+1}$, $B_{2r_0}(x_0) \subset U$. Then for any $\varepsilon > 0$ there exists an open subset $U_1 \subset B_{2r_0}(x_0)$ and $S_{k+1} \cap B_{r_0}(x_0) \subset U_1$, such that*

$$V_{\rho_k}(\partial U_1) < \varepsilon.$$

PROOF. Take a function ζ which is 1 on $B_{r_0}(x_0)$, 0 outside $B_{2r_0}(x_0)$. Then by the Schwartz inequality

$$\int_{B_{2r_0}(x_0)} u_k |\nabla_k(\zeta\psi_k)| \rho_{k+1} \le C,$$

for some constant C that only depends on the minimal slicing.

By the coarea formula this implies that

$$\int_0^\infty dt \int_{\zeta\psi_k = t} u_k \rho_{k+1} d\mu_{k-1} < \infty.$$

Since $\rho_k = u_k \rho_{k+1}$ this means that

$$\int_0^\infty V_{\rho_k}(\zeta\psi_k = t) dt < \infty.$$

Therefore $V_{\rho_k}(\zeta\psi_k = t) < \varepsilon$ for some t. Define $U_1 = \{x : \zeta\psi_k > t\}$. □

REMARK 6.6. Previously we conclude that the rescaled minimal slicing $\Sigma_{k,\sigma}$ converges together with the quadratic form $u_{k,\sigma}$. This volume non-collapsing result implies that the limit $\Sigma_{k,\infty}$ of $\Sigma_{k,\sigma}$, as σ tends to infinity, does not collapse to the singular set $\mathcal{S}_{k+1}$. In fact, for a fixed $\sigma > 0$, take $\varepsilon < \frac{1}{2}\theta\sigma^{k+d}$, where $\theta > 0$ is the weighted volume density of the homoegeneous minimal slicing $\Sigma_{k,\infty}$ at p, and d is the degree of homogeneity of the weight function $\rho_{k,\infty}$ on $\Sigma_{k,\infty}$. Then there exists an open subset U, compactly supported in $B_{2\sigma}(p)$, such that $\mathcal{S}_{k+1} \cap B_\sigma(p) \subset U$, and $V_{\rho_k}(\partial U) < \varepsilon$. Since Σ_k minimizes the weighted volume V_{ρ_k}, we conclude that

$$V_{\rho_k}(\Sigma_k \cap \mathcal{S}_{k+1} \cap B_\sigma(p)) < V_{\rho_k}(\partial U \cap B_\sigma(p)) < \frac{1}{2}\varepsilon\sigma^{k+d}.$$

Letting $\sigma \to 0$, we conclude that in $\Sigma_{k,\infty}$,

$$\mathrm{Vol}(\Sigma_{k,\infty} \cap \mathcal{R}_{k+1} \cap B_1(0)) > \frac{1}{2}\theta > 0.$$

In particular, the limit homogeneous minimal slicing is a k-dimensional slicing.

7. Dimension reduction

We conclude the regularity theory of minimal slicings in this section. Assume $\Sigma_k \subset \cdots \subset \Sigma_n \subset \mathbb{R}^N$ is a minimal k-slicing and $p \in \mathcal{S}_k$. The rescaled minimal slicings at p

$$\Sigma_{k,\sigma} \subset \cdots \subset \Sigma_{n,\sigma}$$

converge subsequencially, as $\sigma \to 0$, to a nontrivial homogeneous minimal slicing in the unit ball of $\mathbb{R}^N$. We analze the singular set $\mathcal{S}_k$ by Federer's dimension reduction argument. Roughly speaking, taking homogeneous minimal slicing at a singular point does reduce the Hausdorff dimension of singular set. On the other hand, a homogeneous minimal slicing has a cone structure and hence its singular set splits off an Euclidean factor $\mathbb{R}^d$. By repeating this process finitely many times the dimension d of the Euclidean factor in the singular set is maximized. We then arrive at a minimal slicing

$$C_k \times \mathbb{R}^d \subset \cdots \subset C_n \times \mathbb{R}^d \subset \mathbb{R}^N$$

such that C_k is a nontrivial minimal cone which is only singular at the origin. The next proposition rules out such a phenomenon for low dimension minimal slicings.

PROPOSITION 7.1. *There is no nontrivial homogeneous minimal 2-slicings with C_2 regular away from 0:*

$$C_2 \subset C_3 \subset \cdots \subset C_{n-1} \subset \mathbb{R}^n.$$

PROOF. Using the unweighted eigenvalue inequality 3.30 we have that

$$\int_{C_2} \left(\frac{3}{4} \sum_{j=3}^{n-1} |\nabla_2 \log u_j|^2 - R_2 \right) \varphi^2 d\mu_2 \leq 4 \int_{C_2} |\nabla_2 \varphi|^2 d\mu_2,$$

for any function $\varphi \in C_0^\infty(C \setminus \{0\})$.

First notice that a two dimension cone is always flat, that is, $R_2 = 0$. Therefore

$$\int_{C_2} \frac{3}{4} \sum_{j=3}^{n-1} |\nabla_2 \log u_j|^2 \varphi^2 d\mu_2 \leq 4 \int_{C_2} |\nabla_2 \varphi|^2 d\mu_2.$$

We apply the logarithmic cut-off trick and send the right hand side to 0. Precisely, define a Lipschitz function on C_2

$$\varphi_{\varepsilon,R}(r) = \begin{cases} 0 & r < \varepsilon^2 \\ \frac{\log \varepsilon^{-2} r}{\log \varepsilon^{-1}} & \varepsilon^2 \leq r \leq \varepsilon \\ 1 & \varepsilon \leq r \leq R \\ \frac{\log R^2 r^{-1}}{\log R} & R \leq r \leq R^2 \\ 0 & R^2 \leq r. \end{cases}$$

Since C_2 has quadratic area growth, namely $\frac{\mathrm{Vol}(C_2 \cap B_r(0))}{\pi r^2} < C$, using the coarea formula we conclude that

$$\int_{C_2} |\nabla_2 \varphi_{\varepsilon,R}|^2 \leq \frac{C}{|\log \varepsilon|} + \frac{C}{\log R} \to 0,$$

as $\varepsilon \to 0$, $R \to \infty$. Therefore $\nabla_2 u_j = 0$ for $j = 3, \cdots, n-1$. That is, each u_j should be a constant function.

Choose an integer $m \geq 3$ such that C_m is the maximal dimensional singular cone. That is, for every integer $j > m$, $C_j = \mathbb{R}^j$, but C_m is singular. By Proposition 5.4 we conclude that u_m is a homogeneous function with negative degree of homogeneity. Contradiction. $\square$

Now we describe the dimension reduction argument.

THEOREM 7.2. *Assume that Σ_{k+1} is partially regular, that is,* $\dim(\mathcal{S}_{k+1}) \leq k-2$. *Then Σ_k is partially regular.*

PROOF. Since $\dim(\mathcal{S}_{k+1}) \leq k-2$, we have that $\dim(\mathcal{S}_k) \leq k-2$. We prove that $\dim(\mathcal{S}_k) \leq k-3$. Suppose, for the sake of contradiction, that $\dim(\mathcal{S}_k) > k-3$. Pick a real number $d \in (k-3, \dim(\mathcal{S}_k))$. Recall that for any compact set A, its outer infinity measure is defined as

$$\mathcal{H}^d_\infty(A) = \{\sum_i r_i^d : A \subset \bigcup B_{r_i}(x_i).\}$$

The important property we will use is that

$$\mathcal{H}^d_\infty(A) > 0 \Leftrightarrow \dim A \geq d.$$

For a sequence of recaled surfaces $\Sigma_k^{(i)} \to \Sigma_k$, their singular sets also converges. Precisely, for any $\varepsilon > 0$, there exists an integer i such that

$$\mathcal{S}(\Sigma_k^{(i)}) \subset N_\varepsilon(\mathcal{S}(\Sigma_k)).$$

Therefore

$$\mathcal{H}^d_\infty(\mathcal{S}(\Sigma_k)) \geq \limsup_i \mathcal{H}^d_\infty(\mathcal{S}(\Sigma_k^{(i)})).$$

We then conclude that for $\mathcal{H}^d_\infty$ almost every x_0 in $\mathcal{S}(\Sigma_k)$ there exists a constant $c = c(d, n) > 0$ such that

$$\limsup_{\sigma\to 0} \frac{\mathcal{H}^d_\infty(\mathcal{S}(\Sigma_k) \cap B_\sigma(x_0))}{\sigma^d} \geq c.$$

Choose a point x_0 with the above lower density bound. Rescaling the surface at x_0 produces a homogeneous minimal slicing

$$C_k \subset \cdots \subset C_{n-1}$$

that satisfies

$$\mathcal{H}^d_\infty(\mathcal{S}(C_k) \cap B_1(0)) \geq c.$$

In particular, if $d > 0$ then $\mathcal{S}(C_k) \cap \partial B_1(0) \neq \emptyset$. Then pick $x_1 \in C_k \cap \partial B_1(0)$ of $\mathcal{H}^d_\infty$ density positive and rescale the minimal slicing at x_1. We then obtain a homogeneous minimal slicing in the form

$$\hat{C}_{k-1} \times \mathbb{R} \subset \hat{C}_k \times \mathbb{R} \subset \cdots$$

Repeat the argument $k - j$ times until we get

$$\tilde{C}_j \times \mathbb{R}^{k-j} \subset \tilde{C}_{j+1} \times \mathbb{R}^{k-j-1} \subset \cdots$$

with each $\tilde{C}_j$ has an isolated singularity at the origin.

Since there is no minimal 2-slicing with C_2 regular away from the origin, we conclude that $j \geq 3$. However, the Hausdorff dimension of $\mathcal{S}(C_j) \times \mathbb{R}^{k-j} = k - j$. Since the Hausdorff dimension does decrease as we perform dimension reduction, $k - j \geq d$. Therefore $d \leq k - j \leq k - 3$, contradiction. □

8. Existence and proof of the main theorem

In this section we develop the existence theory for minimial slicings. It is based on the partial regularity theory in the previous sections. Assume that (M^n, g) is a closed oriented Riemannian manifold which admits a smooth degree 1 map F to the n dimension torus $T^n = S^1 \times \cdots \times S^1$. Take $x^1, \cdots, x^n$ to be the coordinates on each S^1 component. Scale the coordinate functions $x^1, \cdots, x^n$ if necessary, we assume without loss of generality that

$$\int_{S^1} dx^j = 1,$$

on the $j-$th component S^1. Let F^j be the composition of F and the projection from T^n to the j-th component, $j = 1, \cdots, n$:

$$F_j : M^n \to T^n \to S^1.$$

Let $\omega^j = F^*(dx^j) = F_j^*(dx^j)$. Since $\deg F = 1$,

$$\int_M \omega^1 \wedge \cdots \wedge \omega^n = 1.$$

We first describe the construction of the first hypersurface $\Sigma_{n-1} \subset M^n$ in the minimal slicing. To do so let us consider the class of $(n - 1)$ currents

$$\mathcal{C}_{n-1} = \{\Sigma \text{ is an integral } (n-1) \text{ current in } M : \int_\Sigma \omega^1 \wedge \cdots \wedge \omega^{n-1} = 1.\}$$

Then $\mathcal{C}_{n-1}$ is not empty. In fact, at every regular point p of F_n, since F is of degree 1 by the area,

$$\int_{F_n^{-1}(p)} \omega^1 \wedge \cdots \wedge \omega^{n-1} = \int_{F(F_n^{-1}(p))} dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{T^{n-1}} dx^1 \wedge \cdots \wedge dx^{n-1} = 1.$$

Hence by the Sard theorem for almost every $p \in S^1$, $F_n^{-1}(p) \in \mathcal{C}_{n-1}$.

We next minimize mass in the class $\mathcal{C}_{n-1}$, namely consider the variational problem

$$\inf\{M_{n-1}(\Sigma) : \Sigma \in \mathcal{C}_{n-1}.\}$$

The by the compactness theorem for currents with locally uniformly bounded mass the infimum is achieved by some integral $(n-1)$ current Σ_{n-1}. Moreover, since the condition

$$\int_\Sigma \omega^1 \wedge \cdots \wedge \omega^{n-1} = 1$$

is preserved under the convergence of currents $\Sigma_{n-1} \in \mathcal{C}_{n-1}$. Therefore Σ_{n-1} is an area minimizing current and by the usual regularity theory it is regular away from a set of codimension 7.

Assume for the sake of induction that we have constructed

$$\Sigma_{k+1} \subset \cdots \subset \Sigma_{n-1} \subset M$$

with the corresponding first eigenfunctions $u_{k+1}, \cdots, u_{n-1}$ defined on them. To construct Σ_k the most natural idea is to minimize the weighted volume

$$V_{\rho_{k+1}}(\Sigma) = \int_\Sigma \rho_{k+1} d\mu_k,$$

in some class of integral currents. Here μ_k is the k dimensional Hausdorff measure. We require that Σ is an integral current with no boundary on the regular set of Σ_{k+1}. To describe this class of currents precisely, we first need the following

LEMMA 8.1. *Let U be an open subset of T^k with $\bar{U} \neq T^k$, V is an open subset such that $\bar{V} \subset U$. Then there exists a k-form θ_k defined on T^k such that*

$$\theta_k = 0 \text{ in } V, \quad \theta_k = dx^1 \wedge \cdots \wedge dx^k \text{ on } T^k \setminus U,$$

and $dx^1 \wedge \cdots \wedge dx^k - \theta = d\eta$ for some smooth $(k-1)$-form η supported in U.

PROOF. Since U is an open subset of T^k which is not dense, there exists a smooth function f which is identically 1 in U and $\int_{T^k} f dx^1 \cdots dx^k = 0$. Therefore the equation

$$\Delta u = f$$

has a solution u on T^k. Define an $(n-1)$-form η by

$$\eta = \zeta(*du),$$

where ζ is a cut off function which is 1 in V and 0 outside u. We then have

$$\begin{aligned} d\eta &= d * du \\ &= (\Delta u) dx^1 \wedge \cdots \wedge dx^k \\ &= dx^1 \wedge \cdots \wedge dx^k \end{aligned}$$

in V. Finally, let $\theta_k = dx^1 \wedge \cdots \wedge dx^k - d\eta$. It's straightforward to check that

$$\theta_k = 0 \text{ in } V, \quad \theta_k = dx^1 \wedge \cdots \wedge dx^k \text{ in } T^k \setminus U,$$

and that $dx^1 \wedge \cdots \wedge dx^k - \theta_k$ is an exact form supported in U. □

We use θ_k the replacement of the usual volume to construct Σ_k as follows. For each $j = 1, \cdots, n$, denote $F^j : M \to T^j$ the map onto the product of first j S^1 components, and $\theta_{k+1} = 0$ is a form defined as above on T^{k+1} which vanishes in a neighborhood of $T^{k+1}(\mathcal{S}_{k+2})$. Assume by induction that

$$\int_{\Sigma_{k+1}} (F^{k+1})^* \theta_{k+1} = 1,$$

and Σ_{k+1} minimizes the weighted volume $V_{\rho_{k+2}}(\cdot)$.

Choose a k-form θ_k on T^k which vanishes in an open neighborhood U of $F^k(\mathcal{S}_{k+1})$, and is homologous to the usual volume form. We consider the collection of k-currents

$$\mathcal{C}_k = \{\Sigma \text{ locally integrable } k\text{-current in } \mathcal{R}_{k+1}\colon \int_\Sigma (F^k)^*\theta_k = 1.\}$$

Notice that the collection $\mathcal{C}_k$ is non-empty. Consider the map $F_{k+1}|_{\Sigma_{k+1}} : \Sigma_{k+1} \to S^1$. Since F_{k+1} is of degree 1, by the area formula, for almost every point $p \in S^1$,

$$\begin{aligned}\int_{F_{k+1}^{-1}(p)} (F^k)^*\theta_k d\mu_k &= \int_{F(F_{k+1}^{-1}(p))} \theta_k \\ &= \int_{T_k} dx^1 \wedge \cdots \wedge dx^k \\ &= 1.\end{aligned}$$

Since $\rho_{k+1} = u_{k+1}\rho_{k+2}$ and $u_{k+1} \in L^2_{\rho_{k+2}}(\Sigma_{k+1})$, we know that $\int_{\Sigma_{k+1}} \rho_{k+1} d\mu_{k+1} < \infty$. We therefore conclude that

$$\inf\{V_{\rho_{k+1}}(\Sigma) : \Sigma \in \mathcal{C}_k\}$$

is finite. Take any minimizing sequence $\Sigma^{(i)}$ of this variational problem. By the choice of θ_k we conclude that $\Sigma^(i)$ has uniformly bounded mass in $M \setminus U$. On the other hand, by Proposition 6.5, we may choose a small $\varepsilon_0 > 0$ and some neighborhood U of $\mathcal{S}_{k+1}$ such that for sufficiently large i,

$$V_{\rho_{k+1}}(\Sigma^{(i)} \cap U) < \varepsilon_0.$$

Therefore we conclude that the sequence $\Sigma^{(i)}$ has uniformly bounded mass in any compact subset of M. By the usual compactness theorem there is a subsequence converging to a limit Σ_k. Note that Σ_k is also in the class $\mathcal{C}_k$ and it minimizes $V_{\rho_{k+1}}(\cdot)$ in its homology class. We may continue this downward inductive construction until $k = 1$.

Next we describe the construction of the eigenfunction u_k. By the regularity theory developed earlier we know that $\dim \mathcal{S}_k \le k - 3$. Recall that L^2_j is the weighted L^2 space with respect to the weighted measure $\rho_{k+1} d\mu_k$, and that the spaces $\mathcal{H}_k$, $\mathcal{H}_{k,0}$ denote the weighted Sobolev space induced by the norm

$$\|\varphi\|_{1,k} = \int_{\Sigma_k} \varphi^2 \rho_{k+1} d\mu_k + \int_{\Sigma_k} \left(|\nabla_k \varphi|^2 + |A_k|^2 + \sum_{p=k+1}^{n-1} |\nabla_k \log u_p|^2 \right) \varphi^2 \rho_{k+1} d\mu_k,$$

and that we have the following coercivity lemma:

$$\|\varphi\|^2_{k,0} \le c_k(Q_k(\varphi,\varphi) + \|\varphi\|^2_{0,k}).$$

Then we are able to prove

THEOREM 8.2. *There exists an orthonormal basis in $\mathcal{H}_{k,0}$ of eigenfunctions for the quadratic form $Q_k(\cdot,\cdot)$. In particular, there exists a lowest eigenfunction $u_k > 0$. Moreover, the first eigenvalue λ_k is of multiplicity* 1.

PROOF. Using the characterization

$$\lambda_k = \inf\{Q_k(\varphi,\varphi) : \varphi \in \mathcal{H}_{k,0}, \int_{\Sigma_k} \varphi^2 \rho_{k+1} d\mu_k = 1, \}$$

we see that the first eigenvalue λ_k is finite, since $\mathcal{R}_k$ is an open subset of Σ_k. In order to construct the first eigenfunction it suffices to prove a Rellich type lemma. Namely, we prove that

$$\mathcal{H}_{k,0} \hookrightarrow L^2_k$$

is compact.

For any bounded sequence $\{\varphi_i\}$ in $\mathcal{H}_{k,0}$, by the usual Rellich lemma there exists a subsequence that we will also denote by $\{\varphi_i\}$, that converges in $L^2_{\text{loc}}(\mathcal{R}_k)$ to a limit function φ locally in the regular set $\mathcal{R}_k$. To see that the convergence $\varphi_i \to \varphi$ is also in $L^2_k(\Sigma_k)$, we use the L^2 non-concentration Proposition 3.19. We see that for any $\eta > 0$ there exists an open neighborhood V of $\mathcal{S}_{k+1}$ such that

$$\|\varphi_i\|_{L^2_k(V)} \leq \eta \|\varphi_i\|_{k,0}(\Sigma) < \eta C.$$

Hence $\varphi_i \to \varphi$ in $L^2_k(\Sigma_k)$. The existence of an orthonormal basis of $L^2_k(\Sigma_k)$ by eigenfunctions of Q_k is then given by the min-max characterization of eigenvalues.

Since $\mathcal{S}_k$ is of Hausdorff dimension at most $k-3$, the regular set $\mathcal{R}_k$ is connected. Therefore the first eigenfunction is positive on $\mathcal{R}_k$, and that λ_k is of multiplicity 1. □

REMARK 8.3. With the help of the proper function ψ_k defined in the previous section it is easy to check that in the definition of $\mathcal{H}_{k,0}$, it is equivalent to take the closure of functions supported in $\mathcal{R}_k$, or restrict ambient Lipschitz functions on Σ_k. Once again, a similar argument as in the proof of Proposition 5.4 shows that the capacity of the singular set $\mathcal{S}_k$ is zero.

We are now in the position to prove the main theorem.

THEOREM 8.4. *Suppose M^n is a smooth oriented closed manifold which admits a degree* 1 *map onto the torus T^n. Then M does not admit any metric with positive scalar curvature.*

PROOF. Assume the contrary, that M has a metric g with positive scalar curvature. Take a minimal 2-slicing

$$\Sigma_2 \subset \cdots \subset \Sigma_{n-1} \subset M.$$

Since $\dim(\mathcal{S}_j) \leq j-3$ for each j, we conclude that Σ_2 has no singular set. That is, Σ_2 is a smooth oriented surface. Moreover, since $R(g) > 0$, Σ_2 is Yamabe positive. Therefore each component of Σ_2 is a 2-sphere.

On the other hand, by construction

$$\int_{\Sigma_2} (F^2)^*(\theta_2) = 1.$$

Since Σ_2 has no singular set, $\theta_2 = dx^1 \wedge dx^2$. We then conclude that

$$\int_{\Sigma_2} \omega^1 \wedge \omega^2 = 1$$

for some linearly independent closed 1-forms ω^1 and ω^2.

Therefore $H^1(\Sigma_2, \mathbb{R})$ is at least 2 dimensional, and the genus of Σ_2 is at least 1. Contradiction. □

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