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ORIGINAL PAPER

Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$

Henri Anciaux

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Abstract We give new examples of self-shrinking and self-expanding Lagrangian solutions to the Mean Curvature Flow (MCF). These are Lagrangian submanifolds in $\mathbb{C}^n$, which are foliated by $(n-1)$-spheres (or more generally by minimal $(n-1)$-Legendrian submanifolds of $\mathbb{S}^{2n-1}$), and for which the study of the self-similar equation reduces to solving a non-linear Ordinary Differential Equation (ODE). In the self-shrinking case, we get a family of submanifolds generalising in some sense the self-shrinking curves found by Abresch and Langer.

Key words Mean curvature flow · Lagrangian submanifolds · Self-similar

Mathematics Subject Classifications (2000) 35K55 · 58J35 · 53A07

0 Introduction

Self-similar flows arise as special solutions of the mean curvature flow (which we abbreviate as MCF) that preserve the shape of the evolving surface. Analytically speaking, this amounts to making a particular Ansatz in the parabolic Partial Differential Equation (PDE) describing the flow in order to eliminate the time variable and reduce the equation to an elliptic one. In geometric terms, it means looking for curvature properties of a submanifold that make it evolve in the required way.

The study of such explicit, simple examples of flows is hoped to give a better understanding of the behaviour of the flow at and after a singularity. The mean curvature flow is no longer expected to be unique after a singularity has appeared; there are numerical illustrations of this fact (cf. [6]) and current efforts to give a proof of it (cf. [7]).

H. Anciaux (✉)
PUC-Rio, Rua Marques de Sao Vicente,
225 Gavea 22453-900, Rio de Janeiro, RJ, Brasil
e-mail: henri@mat.puc-rio.br

The simplest and most important example of a self-similar flow is when the evolution is a homothety. Such a self-similar submanifold X with mean curvature vector H satisfies the following non-linear, elliptic system:

$$H + \lambda X^{\perp} = 0,$$

where $X^{\perp}$ stands for the projection of the position vector X onto the normal space. If λ is any strictly positive constant, the submanifold shrinks in finite time to a single point under the action of the mean curvature flow, its shape remaining unchanged. If λ is strictly negative, the submanifold will expand, its shape again remaining the same; in this case the submanifold is necessarily non-compact. The case of vanishing λ is the well-known case of a minimal submanifold, which of course is stationary under the action of the flow.

Another interesting property of the MCF is that it preserves the Lagrangian condition, i.e. if the initial data is a Lagrangian submanifold and the flow is smooth, the evolving submanifold remains Lagrangian.

In the present paper, we shall focus on Lagrangian submanifolds of $\mathbb{C}^n$, which are foliated by some $(n-1)$-dimensional leaves, so that the submanifold is completely determined by the data of a planar curve and the self-similar equation reduces to an ODE on it. Analogous methods are used in Angenent et al. [7] and Angenent [5] for problems of the same kind, and in Cao [9] for producing Ricci–Kähler solitons.

All our results are based on the following lemma:

Lemma 1 *Let $\psi\colon \mathcal{M} \to \mathbb{S}^{2n-1}$ be a minimal, Legendrian immersion and $\gamma\colon I \to \mathbb{C}^*$ be a smooth regular curve parametrised by arclength s. Then the following immersion*

$$\begin{aligned} \gamma * \psi\colon\ I \times \mathcal{M} &\to \mathbb{C}^n \\ (s,\sigma) &\mapsto \gamma(s).\phi(\sigma) \end{aligned}$$

is Lagrangian. Moreover, $\gamma * \psi$ satisfies the self-similar MCF equation

$$H + \lambda(\gamma * \psi)^{\perp} = 0$$

for some constant $\lambda \in \mathbb{R}$ if and only if γ satisfies the following equation:

$$k = \langle \gamma, N \rangle \left(\frac{n-1}{|\gamma|^2} - \lambda \right), \tag{1}$$

where k is the curvature of γ and N its unit normal vector.

In the self-shrinking case, we prove the following:

Theorem 1 *For strictly positive λ, Equation 1 admits a countable family of closed solutions, which is parametrised by relatively prime numbers p and q subject to the condition $p/q \in (1/2n, 1/\sqrt{2n})$; p is the winding number of the curve and q is the number of maxima of its curvature. There are no other closed solutions provided that the function $\Phi(E) = \int_{r_-}^{r_+} \frac{\mathrm{d}r}{r} \left(\frac{r^{2n} \exp(-nr^2)}{E} - 1 \right)^{-1/2}$, where r_- and r_+ are the two roots of the equation $r^n \exp(-nr^2/2) = E$, is nondecreasing. To any such curve and any minimal Legendrean immersion $\psi\colon \mathcal{M} \to \mathbb{S}^{2n-1}$ corresponds a Lagrangian immersion $\gamma * \psi$ into $\mathbb{C}^n$, which is self-shrinking.*

We observe that Equation 1 holds even in the special case $n = 1$, where the immersion $\gamma * \psi$ coincides with the curve γ. In particular, we recover all self-shrinking planar curves, which were first found by Abresch and Langer [17] (see also [13]). We conjecture that any compact, Lagrangian self-shrinker of $\mathbb{C}^n$ is the Cartesian product of some of the ones described in Theorem 1.

Equation 1 also turns to be integrable in the case of negative λ. This yields the following:

Theorem 2 *For strictly negative* λ, *Equation* 1 *admits a one-parameter family of embedded curves* $\gamma_t, t \in (0, \pi/n)$ *with two ends asymptotic to two straight lines whose angle is* t. *To any such curve and any minimal Legendrean immersion* $\psi\colon\ \mathcal{M} \to \mathbb{S}^{2n-1}$ *corresponds a Lagrangean immersion*

$$\begin{aligned} \gamma_t * \psi\colon\ \mathbb{R} \times \mathcal{M} &\to \mathbb{C}^n \\ (s, \sigma) &\mapsto \gamma_t(s).\psi(\sigma), \end{aligned}$$

which is self-expanding.

It is well known that the only minimal Legendrean curves into $\mathbb{S}^3$ are great circles, so in dimension 2, all the self-similar surfaces described in Theorems 1 and 2 are *cyclic*, i.e. foliated by round circles. In a forthcoming paper [3], we will give a characterisation of Lagrangean cyclic surfaces. On the other hand, for $n \geq 3$ there are many minimal Legendrean immersions into $\mathbb{S}^{2n-1}$ (cf [8, 11, 15–17]) and we get as many families of self-similar Lagrangian submanifolds.

A very special case,—which has been mentioned first in Chen [12] where they are called *complex extensors*— is when the minimal Legendrean immersion is the totally geodesic embedding

$$\begin{aligned} \psi_0\colon\ \mathbb{S}^{n-1} \subset \mathbb{R}^n &\to \mathbb{S}^{2n-1} \subset \mathbb{C}^n \\ \sigma &\mapsto \sigma + i0. \end{aligned}$$

In particular the immersion $\gamma * \psi_0$ is $\mathrm{SO}(n)$-equivariant for the following action: $x + iy \mapsto Ax + iAy,\ x + iy \in \mathbb{C}^n, A \in \mathrm{SO}(n)$; moreover its image is foliated by round spheres. In a forthcoming paper [2], we will show that, when $n \geq 3$, the only self-similar submanifolds, which are Lagrangian and foliated by $(n-1)$-dimensional round spheres are those obtained in that way.

In that particular case, we may discuss the embeddedness of the self-similar solutions:

Theorem 3 *In the equivariant case the image of a self-shrinking Lagrangian immersion of the form* $\gamma * \psi_0$ *as in Theorem* 1 *is an embedded submanifold if and only if the winding number* p *of the curve* γ *is* 1 *and* q *is even.*

Theorem 4 *In the equivariant case the self-expanding immersions* $\gamma_t * \psi_0$ *described in Theorem* 2 *are embeddings.*

The paper is organised as follows: in the next section, we give the proof of Lemma 1 and in Section 2, we study Equation 1. In Sections 3, 4 and 5, we analyse sucessively the self-similar equation in cases of positive, vanishing and negative λ, deducing Theorems 1, 2, 3 and 4.

I am grateful to Tom Ilmanen for proposing this subject to me, giving me fruitful advice and inviting me to ETH at Zürich, where part of this work was done. I also want to thank Francisco Urbano, Knut Smoczyk and the referee for useful remarks.

1 Proof of lemma 1

An immersion ψ from a manifold $\mathcal{M}$ of dimension $n-1$ into $\mathbb{S}^{2n-1}$ is said to be *Legendrian* if it is horizontal for the distribution defined by the complex structure J, i.e. if $\langle \psi(p), J\psi_* v\rangle = 0, \forall p \in \mathcal{M}, \forall v \in T_p\mathcal{M}$.

It is easy to check that given γ a smooth curve in the complex plane $\mathbb{C}^*$ punctured at 0, the following immersion is Lagrangian:

$$\begin{aligned} \gamma * \psi \colon\ I \times \mathcal{M} &\to \mathbb{C}^n \\ (s,\sigma) &\mapsto \gamma(s).\psi(\sigma). \end{aligned}$$

At some point σ of $\mathcal{M}$ let $(v_1, \dots, v_{n-1})$ be a basis of tangent vectors; then the following family $(\dot\gamma.\psi, \gamma.\psi^* v_1, \dots, \gamma.\psi^* v_{n-1})$ is a basis of tangent vectors to $\gamma * \psi(I \times \mathcal{M})$. We shall express the self-similar equation in the corresponding basis of normal vectors: $N_0 := J\dot\gamma.\psi$ and $N_i := J\gamma.\psi^* v_i, 1 \le i \le n-1$.

We shall use part of a result of Ros and Urbano [18], that we state here in accordance with our notation:

Proposition ([18], Proposition 3, p. 215) *The mean curvature vector of $\gamma * \psi$ at (s,σ) is given by*

$$H = a(s)N_0 + \frac{1}{|\gamma|^2}\gamma.H_\psi,$$

where H_ψ is the mean curvature vector of ψ (viewed as an immersion into $\mathbb{S}^{2n-1}$) at σ, and

$$a(s) = k(s) - (n-1)\frac{\langle \gamma, N\rangle}{|\gamma|^2},$$

where $k(s)$ is the curvature of the γ.

In particular, if we assume that ψ is minimal, then H_ψ vanishes and

$$H = \Big(k(s) - (n-1)\frac{\langle \gamma, N\rangle}{|\gamma|^2}\Big)N_0.$$

We now calculate the normal projection of $\gamma * \psi$:

$$\langle \gamma * \psi, N_i\rangle = \langle \gamma.\psi, J\gamma.\psi^* v_i\rangle = |\gamma|^2 \langle \psi, J\psi^* v_i\rangle = 0,$$

$$\langle \gamma * \psi, N_0\rangle = \langle \gamma.\psi, J\dot\gamma.\psi\rangle = |\psi|^2 \langle \gamma, J\dot\gamma\rangle = \langle \gamma, N\rangle.$$

Hence, we have

$$(\gamma * \psi)^\perp = \langle \gamma, N\rangle N_0,$$

so H and $(\gamma * \psi)^\perp$ are colinear and the image of the immersion $\gamma * \psi$ is self-similar if and only if (1) holds.

2 Solving Equation 1

If we parametrise the curve $\gamma = re^{i\phi}$ by arclength and denote by θ the angle made by the tangent with any fixed axis, Eq. 1 may be written as the following system:

$$\begin{cases} \dot{r} = \cos(\theta - \phi) \\ \dot{\phi} = \frac{1}{r}\sin(\theta - \phi) \\ \dot{\theta} = \left(\lambda r - \frac{n-1}{r}\right)\sin(\theta - \phi). \end{cases}$$

The first two equations of the system express the arclength parametrisation assumption and the last one corresponds directly to (1).

We then introduce the new variable $\alpha := \theta - \phi$, which is more intrinsic than θ and ϕ in the sense that it does not depend on the choice of a reference axis in $\mathbb{C}$. This allows to reduce the previous system to

$$\begin{cases} \dot{r} = \cos\alpha \\ \dot{\alpha} = \left(\lambda r - \frac{n}{r}\right)\sin\alpha. \end{cases} \tag{2}$$

We notice that trivial integral curves to this last system are the half-lines $\{\alpha = 0[\pi]\}$, which correspond to γ being a straight line passing through the origin. As a consequence, other integral curves must stay in a region $\{k\pi < \alpha < (k+1)\pi, k \in \mathbb{Z}\}$ of the half-plane. The system being periodic in α, we may restrict ourselves to $\{0 < \alpha < 2\pi\}$ and even to $\{0 < \alpha < \pi\}$ because of the symmetry of the system.

Finally, we notice that it has a first integral, namely

$$E := r^n \exp(-\lambda r^2/2)\sin\alpha.$$

We have drawn on Figures 1 and 2 the phase portrait, distinguishing the case, $\lambda > 0$ in which the trajectories are bounded, from the case $\lambda \leq 0$, in which they are not.

3 The self-shrinking case (Proof of Theorems 1 and 3)

Without loss of generality, we may fix $\lambda = n$. System (2) has as equilibrium points $(\alpha = \frac{\pi}{2}[\pi], r = 1)$. Geometrically, they correspond to γ being the unit circle. These points are local extrema of the energy E, so integral curves are closed curves around them, except the vertical lines $\{\alpha = 0[\pi]\}$. However, closedness of the integral curves does not imply closedness of the curve γ. This is indeed the case if and only if the total variation of the phase $\Phi(E) := \int \dot{\phi}\, \mathrm{d}s$ (where the integration is computed along the integral curve of level E) equals a rational number p/q times 2π. In this case, by reproducing q times the curve, we get a closed one, that we also call γ. Then the integers p and q have a geometric meaning: p is the winding number of γ and q is the number of maxima of the curvature, which is also the number of patterns (or petals, since the curve looks like a rosette).

If $\Phi(E)$ is not rationally related to 2π, the full curve γ makes infinitely many winds and is dense in some subset of $\mathbb{C}$.

Next, we state the following lemma:

Lemma 2 *The following limits hold*:

(1) $\lim_{E\to 0} \Phi(E) = \pi/n$,

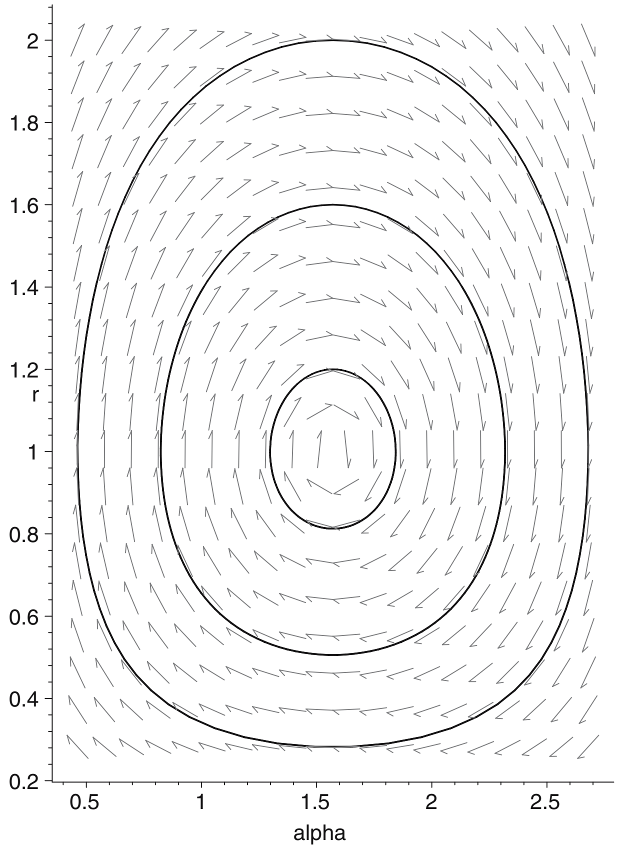

Fig. 1 Phase portrait, $\lambda > 0$

(2) $\lim_{E \to \exp(-n/2)} \Phi(E) = 2\pi/\sqrt{2n}$.

Proof For an integral curve of level E, we shall denote by $r_-(E)$ and $r_+(E)$ the minimal and maximal values taken by r; this corresponds to the intersection of the curve (α, r) with the straight line $\{\alpha = \pi/2\}$, so we have

$$r_-^n \exp(-nr_-^2/2) = r_+^n \exp(-nr_+^2/2).$$

We now introduce the function $e(r) := r^n \exp(-nr^2/2)$, so that $E = e(r_-) = e(r_+)$. We then have the following expressions for Φ:

$$\Phi(E) = 2\int_{r_-}^{r_+} \frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}r = 2\int_{r_-}^{r_+} \frac{\mathrm{d}r}{r}\left(\frac{r^{2n}\exp(-nr^2)}{r_+^{2n}\exp(-nr_+^2)} - 1\right)^{-1/2}$$
$$= 2\int_{r_-}^{r_+} \frac{\mathrm{d}r}{r}\left(\frac{e^2(r)}{E^2} - 1\right)^{-1/2}.$$

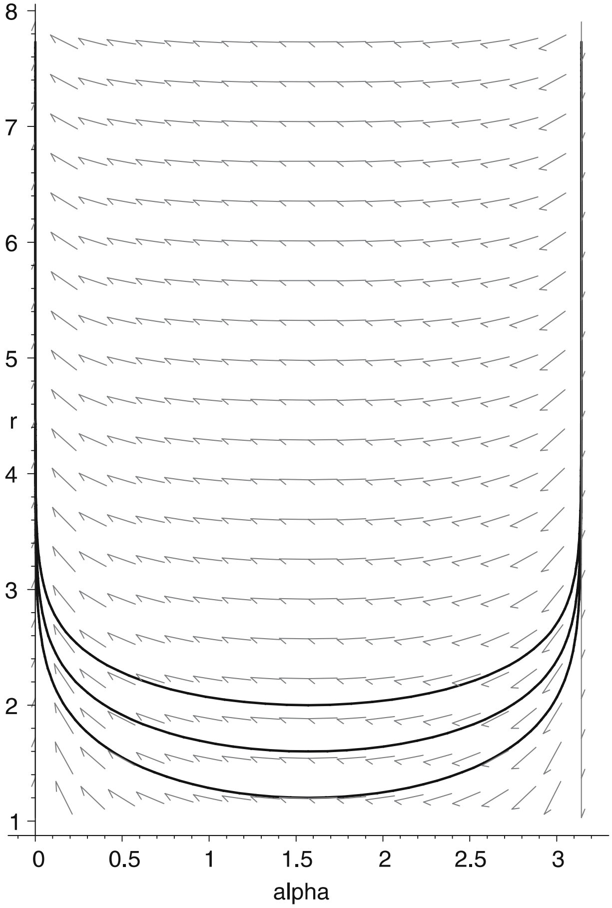

Fig. 2 Phase portrait, $\lambda \le 0$

Proof of (1). We split the integral into two parts: $\Phi = \Phi_1 + \Phi_2 := 2\int_{r_-}^{1} + 2\int_{1}^{r_+}$. We first deal with Φ_1 making the change of variable $x = r/r_-$. We get

$$\Phi_1 = 2\int_1^{1/r_-} \frac{\mathrm{d}x}{x}\left(x^{2n}\exp(nr_-^2(1-x^2)) - 1\right)^{-1/2}.$$

So we deduce that

$$\lim_{E\to 0}\Phi_1(E) = \lim_{r_-\to 0}\Phi_1(r_-) = 2\int_1^{\infty}\frac{\mathrm{d}x}{x}\left(x^{2n}-1\right)^{-1/2} = \frac{\pi}{n}.$$

It remains to show that Φ_2 tends to 0 when r_+ tends to ∞. It is done in a similar manner, using the change of variable $x = \frac{r-1}{r_+-1}$. The details are left to the reader.

Proof of (2). We now compute the limit of $\Phi_1(E)$ when E tends to $\exp(-n/2)$ by making the change of variable $r = 1 + hx$, where we denote $h := 1 - r_-$. So we have

$$\Phi_1(E) = 2\int_{r_-}^{1} \frac{e(r_-)\mathrm{d}r}{r(e^2(r) - e^2(r_-))^{-1/2}}$$
$$= 2\int_{-1}^{0} \frac{e(1-h)\,h\mathrm{d}x}{(1+hx)\big(e^2(1+hx) - e^2(1-h)\big)^{1/2}}.$$

Now, using the Taylor expansion of e at 1:

$$e(1-h) = \exp(-n/2)\big(1 - nh^2 + \mathrm{o}(h^2)\big),$$

we deduce that

$$\frac{e(1-h)\,h}{(1+hx)\big(e^2(1+hx) - e^2(1-h)\big)^{1/2}}$$
$$= \frac{(1-nh^2+\mathrm{o}(h^2))\,h}{(1+hx)\Big(\big(1-n(xh)^2+\mathrm{o}((xh)^2)\big)^2 - \big(1-nh^2+\mathrm{o}(h^2)\big)^2\Big)^{1/2}},$$

which converges uniformly on any interval strictly contained in $(-1,0)$ to $\frac{1}{\sqrt{2n(1-x^2)}}$ as h tends to 0. Hence

$$\lim_{E\to\exp(-n/2)} \Phi_1(E) = \int_{-1}^{0} \frac{\mathrm{d}x}{\sqrt{2n(1-x^2)}} = \frac{\pi}{\sqrt{2n}}.$$

Analogously, we show that $\lim_{E\to\exp(-n/2)} \Phi_2(E) = \lim_{r_+\to 1} \Phi_2(E) = \frac{\pi}{\sqrt{2n}}$.

This lemma implies that the range of $\Phi(E)$ contains $(\pi/n, 2\pi/\sqrt{2n})$. It seems very likely that Φ is nondecreasing and so its range is exactly the interval $(\pi/n, 2\pi/\sqrt{2n})$.

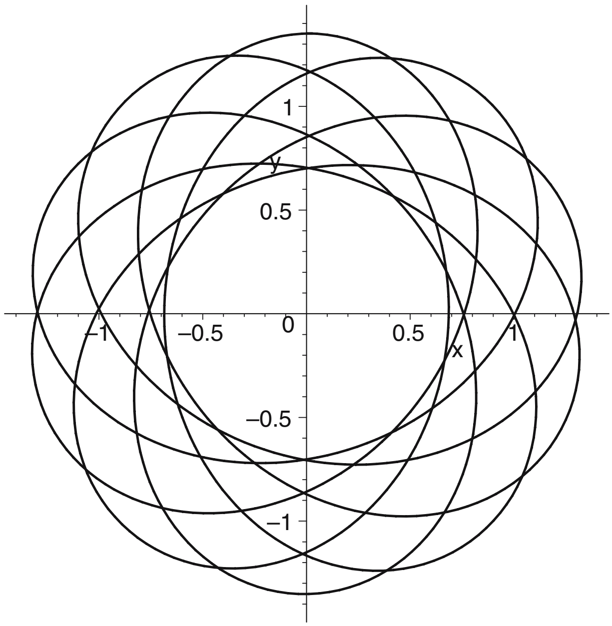

Fig. 3 An Abresch and Langer curve with $p = 7, q = 10$

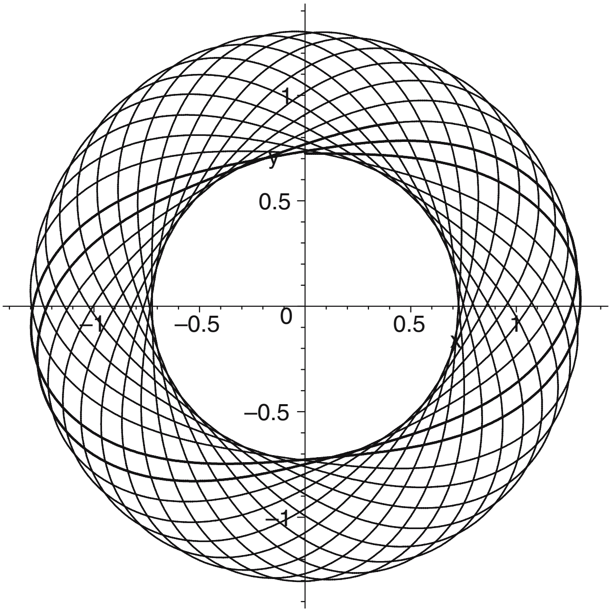

Fig. 4 $n = 2, p = 17, q = 35$

This has been proved for $n = 1$ by Abresch and Langer [1] and recently by Andrews [4]; computer experiments provide some evidence of this fact in higher dimension.

We also notice two important novelties when $n > 1$: first, the curvature of the curve γ is $\dot{\theta} = (nr - \frac{n-1}{r}) \sin\alpha$, which changes sign with the quantity $nr - n - 1/r$. So there is a critical value $E_0 = (\frac{n-1}{n})^{(n-1)/2} \exp(-n - 1/2)$ of E, which has the following property: if $E < E_0$, the trajectory (α, r) never enters the region $\{r < \sqrt{\frac{n-1}{n}}\}$, so the curvature remains strictly positive and γ is (locally) convex. On the other side, if $E > E_0$, a part of the trajectory (α, r) lies in $\{r < \sqrt{\frac{n-1}{n}}\}$ and γ changes convexity. Another fact is that when $n > 1$ the interval $(1/2n, 1/\sqrt{2n})$ contains rationals of the form $1/q$, which correspond to curves γ which are embedded. This is important for the proof of Theorem 3 that we discuss now.

We first observe that a necessary solution for the immersion $\gamma * \psi_0$ to be an embedding is that the curve γ is itself embedded. So necessarily the winding number p of γ must be one, which can occur except in dimension $n = 1$. However this condition is not sufficient for the corresponding submanifold to be embedded: if the curve admits two points $\gamma(s_1)$ and $\gamma(s_2)$ which are symmetric with respect to the origin, we have:

$$\gamma * \psi_0(s_1, \sigma) = \gamma(s_1).\sigma = -\gamma(s_2).\sigma = \gamma * \psi_0(s_2, -\sigma), \quad \forall \sigma \in \mathbb{S}^{n-1}.$$

Since the sphere $\mathbb{S}^{n-1}$ is also symmetric with respect to the origin, it implies that the two spherical leaves corresponding to $\gamma(s_1)$ and $\gamma(s_2)$ are identical. On the other hand, two distinct points of γ which are not symmetric with respect to the origin correspond to two spherical leaves, which do not intersect. Hence, there are two different situations:

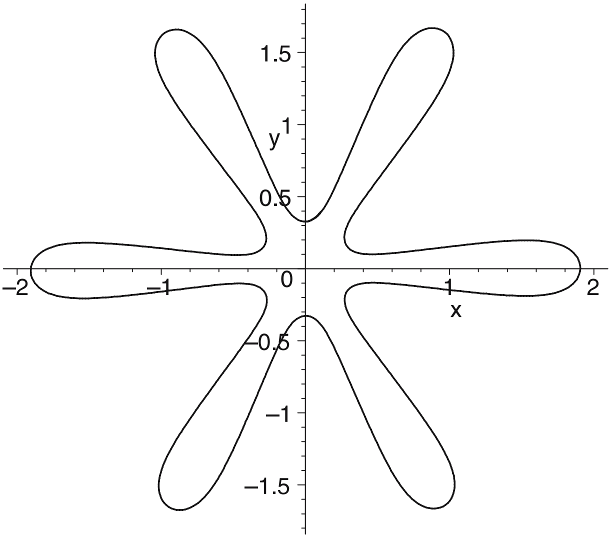

Fig. 5 $n = 5, p = 1, q = 6$

(1) if q is odd, a simple argument on the variations of the radius $r(s)$ and the angle $\phi(s)$ of the curve shows that there are a finite number of pairs of such symmetric points on γ. Thus, the submanifold has self-intersections;
(2) if q is even, the whole curve has a central symmetry, and to any pair $(\gamma_0, -\gamma_0)$ of antipodal points of γ corresponds a unique spherical leaf in the image. It follows that the immersion $\gamma * \psi$ is a two-covering of its image, which is a embedded submanifold, since the quotient of γ by the antipodal map is embedded. For example, in $\mathbb{C}^5$ we get an embedded Lagrangian self-shrinker by taking $(p, q) = (1, 6)$) (See Fig. 5).

4 The minimal case

In this case, we may give explicitly the solutions. Indeed, we have

$$\dot{\phi} = \frac{1}{r} \sin \alpha = -\frac{1}{n}\dot{\alpha},$$

so up to a constant, we have

$$\phi = -\frac{1}{n}\alpha = -\frac{1}{n} \arcsin \frac{E}{r^n}.$$

If $n = 1$, the corresponding curve γ is a straight line and in larger dimensions, we can check that $\text{Im}\,(\gamma^n) = \text{Const}$. In the equivariant case $\gamma * \psi_0$, we recognize the *Lagrangian catenoids*, which were first identified by Harvey and Lawson [14] and was studied in greater depth by Castro and Urbano [10]. The equation $\text{Im}\,(\gamma^n) = \text{Const}$ and its relations with special Lagrangian submanifolds was also considered independently in Haskins [15] and Joyce [16]. The angle ϕ has range $(-\pi/n, 0)$, which corresponds to the fact that the Lagrangian catenoid is asymptotic to two Lagrangian hyperplanes with a constant angle π/n.

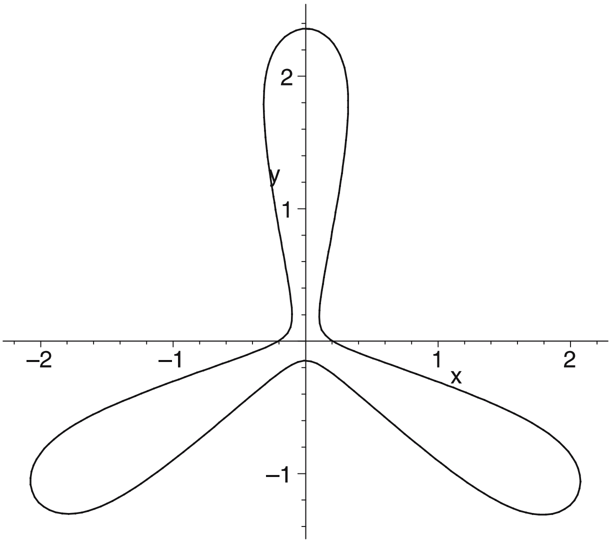

Fig. 6 $n = 2, p = 1, q = 3$

5 The self-expanding case (Proof of Theorems 2 and 4)

From now on, we fix $\lambda = -n$. In the remainder of the section, we shall label any integral curve by the minimal value it takes, denoted by r_0. Notice that the range of r_0 is $(0, \infty)$ and that the energy E is expressed by $E = r_0^n \exp(nr_0^2/2)$.

We now compute the phase variation of γ:

$$\Phi(r_0) := \int \dot{\phi} = 2\int_{r_0}^{\infty} \frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}r = 2\int_{r_0}^{\infty} \frac{\sin\alpha}{r\cos\alpha}\mathrm{d}r$$
$$= 2\int_{r_0}^{\infty} \frac{\mathrm{d}r}{r}\left(\frac{r^{2n}\exp(-nr^2)}{E^2} - 1\right)^{-1/2}$$
$$= 2\int_{r_0}^{\infty} \frac{\mathrm{d}r}{r}\left(\left(\frac{r}{r_0}\right)^{2n}\exp(n(r^2 - r_0^2)) - 1\right)^{-1/2}.$$

Making the change of variable $x = r/r_0$, we get

$$\Phi(r_0) = 2\int_1^{\infty} \frac{\mathrm{d}x}{x}\left(x^{2n}\exp(nr_0^2(x^2-1)) - 1\right)^{-1/2}.$$

From this last expression, standard computations yield that $\Phi(r_0)$ is decreasing and has range $(0, \pi/n)$. The fact that this integral is finite means that γ_t has two ends asymptotic to two straight lines, whose angle is just the value of the integral. Thus the corresponding self-expander has two ends, which, in the equivariant case $\gamma_t * \psi_0$, are asymptotic to two Lagrangian subspaces. Finally we observe that the fact that the angle function ϕ of γ_t is monotone and has range $(0, \pi/n)$ implies that γ_t is embedded and contained in a half-space. By a similar argument to that of Section 3, we conclude that $\gamma_t * \psi_0$ is an embedding.

References

1. Abresch, U., Langer, J.: The normalized curve shortening flow and homothetic solutions. J. Differential Geom. **23**, 175–196 (1986)
2. Anciaux, H., Castro, I., Romon, P.: Lagrangian submanifolds foliated by $(n-1)$-spheres in $\mathbb{R}^{2n}$. to appear in Acta Math. Sinica
3. Anciaux, H., Romon, P.: Cyclic and ruled Lagrangian surfaces in complex Euclidean space. In preparation
4. Andrews, B.: Classification of limiting shapes for isotropic curve flows. J. Amer. Math. Soc. **16**(2), 443–459 (2003)
5. Angenent, S.: Shrinking donuts. In: Lloyd, N.G. (ed.), Nonlinear diffusion reaction equations & their equilibrium, States 3. Birkhäuser, Boston (1992)
6. Angenent, S., Ilmanen, T., Chopp, D.L.: A computed example of non-uniqueness of mean curvature flow in $\mathbf{R}^3$. Comm. Partial Differential Equations **20**(11–12), 1937–1958 (1995)
7. Angenent, S., Ilmanen, T., Velázquez, J.J.L.: Fattening from smooth initial data in mean curvature flow. in preparation
8. Borrelli, V., Gorodski, C.: Minimal Legendrian submanifolds of $\mathbb{S}^{2n-1}$ and absolutely area-minimizing cones. Differential Geom. Appl. **21**(3), 337–347 (2004)
9. Cao, H.-D.: Existence of Gradient Kähler-Ricci solitons. In: Chow, B., Gulliver, R., Levy, S., Sullivan, J. (eds.) Elliptic and Parabolic Methods in Geometry. Proceedings of a workshop, Minneapolis, MN, USA, May 23–27, 1994 pp. 1–16. A K Peters, Wellesley, USA (1996)
10. Castro, I., Urbano, F.: On a Minimal Lagrangian Submanifold of $\mathbb{C}^n$ Foliated by Spheres. Mich. Math. J. **46**, 71–82 (1999)
11. Carberry, E., McIntosh, I.: Special Lagrangian tori in $\mathbb{C}P^2$ come in real families of every dimension. J. London Math. Soc. **69**(2), 531–544 (2004)
12. Chen, B.-Y.: Complex extensors and Lagrangian submanifolds in complex Euclidean spaces. Tohoku Math. J. **49**, 277–297 (1997)
13. Epstein, C.L., Weinstein, M.I.: A stable manifold theorem for the curve shortening equation. Comm. Pure Appl. Math. **40**, 119–139 (1987)
14. Harvey, R., Lawson, H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)
15. Haskins, M.: Special Lagrangian cones. Amer. J. Math. **126**, 845–871 (2004)
16. Joyce, D.: Special Lagrangian m-folds in $\mathbb{C}^m$ with symmetries. Duke Math. J. **115**(1), 1–51 (2002)
17. McIntosh, I.: Special Lagrangian cones in $\mathbb{C}^3$ and primitive harmonic maps. J. London Math. Soc. **67**(3), 769–789 (2003)
18. Ros, A., Urbano, F.: Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere. J. Math. Soc. Japan **50**(1), 203–226 (1998)
19. Smoczyk, K.: The Lagrangian mean curvature flow. Habilitationsschrift (English with German preface), University of Leipzig, p. 102 (1999) available at `http://personal-homepages.mis.mpg.de/smoczyk/`
20. Wang, M.-T.: Mean Curvature Flows in Higher Codimension. In: Proceedings of International Congress of Chinese Mathematicians (2001)